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EXTENDING REDHEFFER'S MATRIX
TO ARBITRARY ARITHMETIC FUNCTIONS

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ABSTRACT

The class of Redheffer matrices are distinctive for having determinants equal to the Mertens function. We describe an embedding of the arithmetic functions into the general linear group which allows a generalization of Redheffer's matrices. This generalization exhibits determinants equal to the sum of the Dirichlet convolution inverse of a given invertible arithmetic function, and allows a more general analysis of the mechanisms at work behind Redheffer's original matrices. Following past work by Robert Vaughan, we conduct a basic but general analysis of the Eigenvalues of these Redheffer-type matrices, and we conduct a more in-depth analysis on the matrices corresponding to non-principal Dirichlet characters. We additionally discuss an alternate geometric bound on an important class of coefficients encountered during the analysis, and present a natural generalization of the Eulerian polynomials which emerges when working with certain convolution inverses.

Contents

1	Introduction	1
2	An Embedding of Arithmetic Functions in $GL(l, \mathbb{C})$	1
3	The Characteristic Polynomial of $\rho_l(f)$	4
4	Analysis on Non-principal Dirichlet Characters	6
5	An Alternate Geometric Bound for $S_k(l)$	8
6	A Natural Extension of the Eulerian Polynomials	10

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1 Introduction

In 1977, Raymond Redheffer studied a novel sequence (R_l) of $l \times l$ matrices, now known as the Redheffer matrices, defined by $R_l = (r_{i,j})$, with

$$r_{i,j} = \begin{cases} 1 & i \mid j \text{ or } j = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

In [2], Redheffer showed that

$$\det R_l = M(l), \quad (2)$$

where M is the Mertens function

$$M(x) = \sum_{m \leq x} \mu(m), \quad (3)$$

and μ is the classical Möbius function. That is to say, the Redheffer matrices draw a connection between the fundamentally number theoretic Mertens function and the area of linear algebra. Redheffer and others in particular have uncovered the following pattern, here adapted from Robert Vaughan's exposition in [3]:

Theorem 1.1. *The matrix R_l has three classes of eigenvalues in the following proportions.*

- $l - (\lfloor \log_2 l \rfloor + 1)$ “trivial” eigenvalues $\lambda = 1$.
- $\lfloor \log_2 l \rfloor - 1$ “non-trivial” eigenvalues λ with

$$|\lambda - 1| \leq \frac{\log l}{\log 2} + c \frac{(\log l)^3}{l}.$$

- 2 “dominant” eigenvalues λ_{\pm} with

$$\left| \lambda_+ - \sqrt{l} \right| \leq c \log l, \quad \left| \lambda_- + \sqrt{l} \right| \leq c \log l.$$

More precise asymptotic bounds than these have been determined for both the non-trivial and the dominant eigenvalues, and numerical evidence in fact suggests that the non-trivial eigenvalues each lie within the unit disc $\mathcal{D} = \{z : |z| < 1\}$. Proof of the latter fact would be sufficient to demonstrate the validity of the Riemann Hypothesis, but not unexpectedly such a proof has proven to be elusive.

To better understand the mechanisms which cause the Mertens function to appear in relation to this particular class of matrices, Vaughan suggested to the author to investigate the potential of a generalization of the Redheffer matrices which relates more broadly to sums of arithmetic functions. In fact such a generalization does exist, and the present exposition aims to describe the construction and present some analysis along the lines of Vaughan's analysis in [3], but in the more generalized context. Additionally some related topics are presented which branch out from notions in that analysis.

2 An Embedding of Arithmetic Functions in $GL(l, \mathbb{C})$

The starting point of generalizing Redheffer's matrices lies in an interesting way of embedding arithmetic functions into the general linear group. In order to do this, we make use of so-called finite truncations of arithmetic functions, which are put on solid footing by the following Definitions 2.1 to 2.3 and Proposition 2.4.

Definition 2.1. Denote by \mathcal{A} the group of arithmetic functions $f : \mathbb{N} \rightarrow \mathbb{C}$ such that $f(1) \neq 0$, with multiplication $*$ defined for $f, g \in \mathcal{A}$ by

$$f * g(n) := \sum_{uv=n} f(u)g(v).$$

Denote by \mathcal{M} the subgroup of \mathcal{A} consisting of functions $f \in \mathcal{A}$ which are multiplicative.

Definition 2.2. Let $l \in \mathbb{N}$ and let $f : \mathbb{N} \rightarrow \mathbb{C}$. Denote by T_l the operator which maps f onto its restriction to the domain $\{1, \dots, l\}$. We will call this operator the l -truncation operator, and for any arithmetic function f , we will call $T_l f$ the l -truncation, or just the truncation, of f .

Definition 2.3. Let $l \in \mathbb{N}$. Denote by \mathcal{A}_l the set of functions $f : \{1, \dots, l\} \rightarrow \mathbb{C}$ such that $f(1) \neq 0$. $f \in \mathcal{A}_l$ is said to be multiplicative if $f(ab) = f(a)f(b)$ whenever $ab \leq l$ and a, b are relatively prime. Denote by \mathcal{M}_l the set of functions in \mathcal{A}_l which are multiplicative.

Proposition 2.4. \mathcal{A}_l forms an abelian group with multiplication $*$ defined for $f, g \in \mathcal{A}_l$ by

$$f * g(n) := \sum_{uv=n} f(u)g(v).$$

\mathcal{M}_l forms a (strict) subgroup of \mathcal{A}_l , and T_l is a surjective homomorphism from \mathcal{A} to \mathcal{A}_l which maps \mathcal{M} to \mathcal{M}_l .

Proof. First we show that $T_l(\mathcal{A}) = \mathcal{A}_l$ and $T_l(\mathcal{M}) = \mathcal{M}_l$. To see that $T_l(\mathcal{A}) \supseteq \mathcal{A}_l$ and $T_l(\mathcal{M}) \supseteq \mathcal{M}_l$, suppose that $f \in \mathcal{A}_l$. We construct a function $f_0 \in \mathcal{A}$ so that $T_l f_0 = f$, and so that $f_0 \in \mathcal{M}$ when $f \in \mathcal{M}_l$. First we define γ_f for any prime power p^k , $k \geq 1$ by

$$\gamma_f(p^k) := \begin{cases} f(p^k) & \text{if } p^k \leq l \\ 0 & \text{otherwise} \end{cases}.$$

Then for $n = p_1^{k_1} \dots p_m^{k_m}$ define

$$f_0(n) := \begin{cases} f(n) & n \leq l \\ \prod_{i=1}^m \gamma_f(p_i^{k_i}) & n > l \end{cases}. \quad (4)$$

The first line of this definition ensures that $T_l f_0 = f$, while the second line ensures that f_0 is multiplicative when f is, by incorporating the values of f at prime powers into the definition of f_0 . This is sufficient to prove the inclusion.

The reverse inclusion is simpler to see. $T_l(\mathcal{A}) \subseteq \mathcal{A}_l$ because $T_l f(1)$ is defined for each $l \geq 1$ and is equal to $f(1) \neq 0$, and $T_l(\mathcal{M}) \subseteq \mathcal{M}$ because multiplicativity is preserved by truncation. Thus we see that the images of \mathcal{A} and \mathcal{M} under truncation are \mathcal{A}_l and \mathcal{M}_l respectively.

To see that \mathcal{A}_l forms an abelian group with \mathcal{M}_l as a subgroup, we take advantage of the known group properties of \mathcal{A} by using the observation that for any $f, g \in \mathcal{A}$, we have

$$T_l(f * g)(n) = (T_l f * T_l g)(n).$$

Indeed, this is straightforward to see because the multiplications of \mathcal{A} and \mathcal{A}_l are defined in exactly the same way (over different domains), and because the product $f * g(n)$ only makes use of the values $f(u)$ and $g(v)$ for $u, v \leq n$. Notice that once we have proven that \mathcal{A}_l is a group, this will be sufficient to show that T_l is a group homomorphism with the desired properties.

We can lift the functions in \mathcal{A}_l to the group \mathcal{A} using the previous construction to prove the group axioms on \mathcal{A}_l . To see that \mathcal{A}_l is associative, for instance, let $f, g, h \in \mathcal{A}_l$ with $f_0, g_0, h_0 \in \mathcal{A}$ constructed as above. We have

$$\begin{aligned} (f * g) * h &= (T_l f_0 * T_l g_0) * T_l h_0 = T_l((f_0 * g_0) * h_0) \\ &= T_l(f_0 * (g_0 * h_0)) = T_l f_0 * (T_l g_0 * T_l h_0) = f * (g * h). \end{aligned}$$

Similar arguments give commutativity, the existence of an identity function, and the existence of inverses, so we have that \mathcal{A}_l is a commutative group as claimed. Closure of \mathcal{M}_l under $*$ follows from the fact that $T_l(\mathcal{M}) = \mathcal{M}_l$, and the inclusion of \mathcal{M}_l in \mathcal{A}_l is strict because each function $f \in \mathcal{M}_l$ must have $f(1) = 1$, so any function $g \in \mathcal{A}_l$ with, say, $g(1) = 2$ is not a member of \mathcal{M}_l . \square

So we see that there is a natural correspondence between \mathcal{A} and \mathcal{A}_l and between \mathcal{M} and \mathcal{M}_l . In fact, we show in the following theorem that \mathcal{A}_l and \mathcal{M}_l can readily be described as a matrix group.

Theorem 2.5. \mathcal{A}_l is isomorphic to a subgroup of $\text{GL}(l, \mathbb{C})$, and \mathcal{M}_l is isomorphic to a subgroup of $\text{SL}(l, \mathbb{C})$.

Proof. We construct an explicit isomorphism. For $f \in \mathcal{A}_l$, and define $\varphi_l : \mathcal{A}_l \rightarrow \text{GL}(l, \mathbb{C})$ by $\varphi_l(f) := (\alpha_{i,j}(f))$ where

$$\alpha_{i,j}(f) = \begin{cases} f(j/i) & i \mid j \\ 0 & i \nmid j \end{cases}. \quad (5)$$

To prove that $\varphi_l(\mathcal{A}_l)$ is a subgroup of $\text{GL}(l, \mathbb{C})$, and that φ_l is an isomorphism onto its image, it is sufficient to prove that φ_l is a homomorphism to $\text{GL}(l, \mathbb{C})$ with trivial kernel. Indeed, for $f \in \mathcal{A}_l$, we see that $\varphi_l(f)$ is upper-diagonal with diagonal entries identically equal to $f(1)$, which implies in particular that the matrix is invertible, because it has determinant equal to $f(1)^l \neq 0$.

To see that φ_l is a homomorphism, we let $f, g \in \mathcal{A}_l$ and calculate $\varphi_l(f) \cdot \varphi_l(g) = (a_{i,j})$ with

$$a_{i,j} = \sum_{k=1}^l \alpha_{i,k}(f) \alpha_{k,j}(g) = \begin{cases} \sum_{uv=j/i} f(u)g(v) & i \mid j \\ 0 & i \nmid j \end{cases}.$$

Noticing that this last expression is just $\alpha_{i,j}(f * g)$, we conclude that $\varphi_l(f) \cdot \varphi_l(g) = \varphi_l(f * g)$, and so φ_l is a homomorphism.

From the definition of φ_l it is clear that the kernel of φ_l is just $T_l \mathbf{u}$, where \mathbf{u} is the identity element in \mathcal{A} , so we see that φ_l is a homomorphism into its image, and hence is an isomorphism.

Additionally we notice that for $f \in \mathcal{M}_l$, the diagonal entries of $\varphi_l(f)$ are identically equal to $f(1) = 1$, which implies that the matrix has determinant equal to $f(1)^l = 1$. Thus we also see that $\varphi_l(\mathcal{M}_l) \subseteq \text{SL}(l, \mathbb{C})$. \square

This embedding suggests a natural method of extending Redheffer's matrices to arbitrary functions in \mathcal{A} , described by Definition 2.6.

Definition 2.6. Let $f \in \mathcal{A}$, and define the $l \times l$ matrix $\rho_l(f) := (\beta_{i,j}(f))$ where

$$\beta_{i,j}(f) = \begin{cases} f(j/i) & i \mid j \\ 1 & j = 1, i > 1 \\ 0 & \text{otherwise} \end{cases}.$$

We call $\rho_l(f)$ the $l \times l$ Redheffer-type matrix of f .

We see that in particular, $\rho_l(\mathbf{1})$ is equal to the original $l \times l$ Redheffer matrix R_l . The connection between the Redheffer matrices and the Mertens function is generalized by Theorem 2.7:

Theorem 2.7. *Let $f \in \mathcal{A}$. Then*

$$\det \rho_l(f) = f(1)^l \left(1 + \sum_{k=2}^l f^{-1}(k) \right).$$

Proof. Pre-multiply $\rho_l(f)$ on the left by $\varphi_l(f^{-1})$, which has determinant $f(1)^{-l}$. We have

$$\varphi_l(f^{-1})\rho_l(f) = \varphi_l(f^{-1})(\varphi_l(f) + A) = Id + \varphi_l(f^{-1})A,$$

where $A = (a_{i,j})$ with

$$a_{i,j} = \begin{cases} 1 & j = 1, i > 1 \\ 0 & \text{otherwise} \end{cases},$$

and in particular, $\varphi_l(f^{-1})A$ has a single non-zero entry in its upper triangle, which is in the upper-left corner, and which is equal to $\sum_{k=2}^l f^{-1}(k)$. Thus $\varphi_l(f^{-1})\rho_l(f)$ is in fact diagonal, and a straightforward calculation proves the result. \square

As a special case, if $f(1) = 1$, or in particular if $f \in \mathcal{M}$, then the previous theorem implies that

$$\det \rho_l(f) = \sum_{k=1}^l f^{-1}(k). \tag{6}$$

3 The Characteristic Polynomial of $\rho_l(f)$

We now conduct analysis analogous to that of Vaughan in [3], now in the context of Redheffer-type matrices of a Dirichlet-invertible function $f \in \mathcal{A}$. For technical reasons we require additionally that the Dirichlet series L_f corresponding to f have non-empty domain of convergence.

We start by calculating the characteristic polynomial $p_l^f(\lambda) = \det(\lambda \cdot Id - \rho_l(f))$ of $\rho_l(f)$. Define

$$f_\lambda := f - \lambda \mathbf{u}, \tag{7}$$

where \mathbf{u} is the identity function under convolution. Then for $\lambda \neq f(1)$, $f_\lambda \in \mathcal{A}$, and $p_l^f(\lambda) = \det(-\rho_l(f_\lambda))$ can be calculated via Theorem 2.7 to find

$$p_l^f(\lambda) = (-1)^l \det(\rho_l(f_\lambda)) = (\lambda - f(1))^l \left(1 + \sum_{k=2}^l f_\lambda^{-1}(k) \right). \tag{8}$$

To allow for a simpler calculation of the convolution inverse in this representation, we use a technique employed by Vaughan to normalize f_λ before inversion. Denote $w = f(1)/(\lambda - f(1))$, and define the normalized function g_λ by

$$g_\lambda := -w f_\lambda = (1 + w)f(1)\mathbf{u} - w f.$$

We have in particular that $g_\lambda^{-1} = (-1/w)f_\lambda^{-1}$, which gives

$$p_l^f(\lambda) = (\lambda - f(1))^l - f(1)(\lambda - f(1))^{l-1} \sum_{k=2}^l g_\lambda^{-1}(k). \tag{9}$$

To compute g_λ^{-1} , we observe that the Dirichlet series corresponding to g_λ is given by

$$L_{g_\lambda}(s) = (1 + w)f(1) - wL_f(s) = (1 - w(L_f(s) - f(1))).$$

For $\operatorname{Re}(s)$ sufficiently large, $L_f(s)$ is absolutely convergent and $|L_f(s) - f(1)| < |1/w|$. So for such s we have

$$\begin{aligned} L_{g_\lambda^{-1}}(s) &= (1 - w(L_f(s) - f(1)))^{-1} \\ &= \sum_{k=1}^{\infty} w^k (L_f(s) - f(1))^k = 1 + \sum_{m=2}^{\infty} m^{-s} \sum_{k=1}^{\lfloor \log_2 m \rfloor} w^k D_k^f(m), \end{aligned}$$

where

$$D_k^f(m) := \sum_{\substack{m_1 \cdots m_k = m \\ m_j \geq 2}} \left(\prod_{i=1}^k f(m_i) \right). \quad (10)$$

Thus we find that for $m \geq 2$,

$$g_\lambda^{-1}(m) = \sum_{k=1}^{\lfloor \log_2 m \rfloor} f(1)^k (\lambda - f(1))^{-k} D_k^f(m),$$

and the characteristic polynomial we are looking for is given by

$$p_l^f(\lambda) = (\lambda - f(1))^l - \sum_{k=1}^{\lfloor \log_2 l \rfloor} (\lambda - f(1))^{l-1-k} \left(f(1)^{k+1} S_k^f(l) \right),$$

with

$$S_k^f(l) := \sum_{m=1}^l D_k^f(m). \quad (11)$$

Rewriting this, we have

$$p_l^f(\lambda) = (\lambda - f(1))^{l-N} P_l^f(\lambda),$$

where

$$P_l^f(\lambda) = (\lambda - f(1))^N - \sum_{k=1}^L (\lambda - f(1))^{L-k} S_k^f(l), \quad (12)$$

$$L = \lfloor \log_2 l \rfloor, N = L + 1. \quad (13)$$

In particular, we have shown the following.

Theorem 3.1. *For $f \in \mathcal{A}$ with Dirichlet series convergent in a non-empty domain, $\rho_l(f)$ has at least $l - (\lfloor \log_2 l \rfloor + 1)$ of its eigenvalues equal to $f(1)$.*

4 Analysis on Non-principal Dirichlet Characters

We now consider the more specific case where the function f is chosen to be a non-principal Dirichlet character $\chi \bmod q$. In particular we prove the following result on the eigenvalues of $\rho_l(\chi)$.

Theorem 4.1. *Let χ be a non-principal Dirichlet character. Then $\rho_l(\chi)$ has at least $l - (\lfloor \log_2 l \rfloor + 1)$ “trivial” eigenvalues $\lambda = 1$, and for sufficiently large l , each other eigenvalue satisfies*

$$|\lambda - 1| < (l \log l)^{1/3} + \log l.$$

This contrasts with Vaughan’s findings in [3] on the eigenvalues of $\rho_l(1)$ in that in Vaughan’s case, two eigenvalues dominated with magnitudes close to $\pm\sqrt{l}$, while logarithmic bounds held on all other eigenvalues. In the case of non-principal characters, we see that there are no dominant eigenvalues with magnitude on the order of \sqrt{l} .

A first step in proving this theorem is to notice a simplification that can be made with the coefficients D_k^χ . Because Dirichlet characters are totally multiplicative, we have

$$D_k^\chi(m) = \sum_{\substack{m_1 \cdots m_k = m \\ m_j \geq 2}} \left(\prod_{i=1}^k \chi(m_i) \right) = \chi(m) \cdot \sum_{\substack{m_1 \cdots m_k = m \\ m_j \geq 2}} 1 = \chi(m) D_k^1(m),$$

and as a result,

$$|S_k^\chi(l)| \leq \sum_{m=1}^l |\chi(m)| D_k^1(m) \leq \sum_{m=1}^l D_k^1(m) = S_k^1(l).$$

Denoting $D_k(m) := D_k^1(m)$, and $S_k(m) := S_k^1(m)$, these points can be rewritten as $D_k^\chi(m) = \chi(m) D_k(m)$, and $|S_k^\chi(l)| \leq S_k(l)$.

In [3], Vaughan demonstrates the following bound on the coefficients $S_k(l)$, which we will use here without proof.

Lemma 4.2. *Suppose that k is a natural number and $l \geq 1$. Then*

$$S_k(l) \leq \frac{l(\log l)^{k-1}}{(k-1)!}.$$

We now partition the nontrivial portion of the characteristic polynomial $p_l^f(\lambda)$ into two significant portions by defining polynomials f and g by

$$f(z) = z^N, \quad g(z) = \sum_{k=1}^L z^{L-k} S_k^\chi(l),$$

noting that $P_l^\chi(\lambda) = f(\lambda - 1) - g(\lambda - 1)$. Then using the bound on $S_k(l)$, we see that

$$|g(z)| < |z|^{N-2} \left(|S_1^\chi(l)| + n \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left(\frac{\log(l)}{|z|} \right)^{k-1} \right),$$

or that

$$|g(z)| < |z|^{N-2} \left(|S_1^\chi(l)| + n \left(\exp \left(\frac{\log n}{|z|} \right) - 1 \right) \right). \quad (14)$$

We leave the term $S_1^X(l)$ out of the simplified estimate because we can calculate its value directly, mainly

$$|S_1^X(l)| = \left| \sum_{m=1}^l D_1^X(m) \right| = \left| \sum_{m=1}^l \chi(m) \right| \leq C_q,$$

where C_q is constant with respect to l , with $C_q = O(\sqrt{q} \log q)$ by the Pólya-Vinogradov inequality. Then if ρ satisfies

$$l \left(\exp \left(\frac{\log l}{\rho} \right) - 1 \right) + C_q < \rho^2 \quad (15)$$

and $|z| = \rho$, then

$$|g(z)| < |f(z)|,$$

and this implies by Rouché's theorem that $f(z)$ and $f(z) - g(z)$ have the same number of zeros in the open disc $|z| < \rho$. The following lemma provides a bound on the values of ρ which satisfy inequality (15).

Lemma 4.3. *For $l \geq \max(C_q^3/8, e)$, if*

$$\rho \geq (l \log l)^{1/3} + \log l,$$

then ρ satisfies inequality (15).

Proof. The crux of the proof is the following inequality, which holds for $l > 1$.

$$\frac{\log l}{\log \left(1 + \frac{(l \log l)^{2/3}}{l} \right)} < (l \log l)^{1/3} + \log l. \quad (16)$$

This inequality follows by rearranging the left side to form a difference quotient, and estimating using the mean value theorem. First

$$\frac{\log l}{\log \left(1 + \frac{(l \log l)^{2/3}}{l} \right)} = \frac{\log l}{\log(l + (l \log l)^{2/3}) - \log(l)} = \frac{\log l}{(l \log l)^{2/3} \Delta(l)},$$

where

$$\Delta(l) := \frac{\log(l + (l \log l)^{2/3}) - \log(l)}{(l \log l)^{2/3}}. \quad (17)$$

The mean value theorem gives us that $\Delta(l) = \left(\frac{d}{dx} \log x \right) |^{x=c} = 1/c$ for some $c \in (l, l + (l \log l)^{2/3})$, so in particular

$$\Delta(l) > \frac{1}{l + (l \log l)^{2/3}}.$$

Applying this to our original expression, we have

$$\begin{aligned} \frac{\log l}{\log \left(1 + \frac{(l \log l)^{2/3}}{l} \right)} &= \frac{\log l}{(l \log l)^{2/3} \Delta(l)} \\ &< \frac{(\log l)(l + (l \log l)^{2/3})}{(l \log l)^{2/3}} = (l \log l)^{1/3} + \log l, \end{aligned}$$

demonstrating inequality (16). Thus if $\rho \geq (l \log l)^{1/3} + \log l$, then $\rho > \log l / \log(1 + (l \log l)^{2/3}/l)$, and we have that

$$\rho^2 \geq l^{2/3}(\log l)^{2/3} + 2l^{1/3}(\log l)^{4/3} + (\log l)^2 \quad (18)$$

and

$$l \left(\exp \left(\frac{\log l}{\rho} \right) - 1 \right) + C_q < l^{2/3}(\log l)^{2/3} + C_q, \quad (19)$$

and from these the lemma follows directly. \square

This lemma implies that for large enough l , all N complex roots of the degree N polynomial $f(z) - g(z)$ are contained in the domain $|z| < (l \log l)^{1/3} + \log l$. Using the characterization of $\rho_l(\chi)$ derived in general in section 3, theorem 4.1 follows directly from this analysis.

5 An Alternate Geometric Bound for $S_k(l)$

In the process of investigating bounds on the size of $S_k(l)$, geometric interpretations of the coefficients immediately stood out as promising avenues of attack, and inquiry along these lines produced a bound mainly similar, but slightly distinct, to that employed by Vaughan in [3].

Theorem 5.1. *Suppose that $k \in \mathbb{N}$ and $l \in \mathbb{R}$ with $l \geq 1$. Then*

$$S_k(l) < \frac{l(\log l)^k}{k!}.$$

The proof of this bound relies on the following interesting calculation of the volume of a k -dimensional solid.

Lemma 5.2. *Suppose that $k \in \mathbb{N}$ and $l \in \mathbb{R}$ with $l \geq 1$. Define the k -dimensional solid*

$$H(k, l) := \left\{ (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 \cdots x_k \leq l \text{ and } x_1, \dots, x_k \geq 1 \right\},$$

and let $V_k(l) := \int_{H(k, l)} dV$ be the volume of $H(k, l)$. Then we have

$$V_k(l) = (-1)^k \cdot \left(1 - l \cdot \sum_{i=0}^{k-1} \frac{(-\log l)^i}{i!} \right),$$

and moreover, for each $k \geq 1$ (taking $V_0(l) = 1$ for each l)

$$V_k(l) = \frac{l(\log l)^{k-1}}{(k-1)!} - V_{k-1}(l).$$

Proof. We proceed by induction on k . For $k = 1$, the calculation is straightforward:

$$V_1(l) = \int_{\substack{x_1 \leq l \\ x_1 \geq 1}} dV = \int_1^l dx_1 = l - 1 = (-1)^1 \cdot \left(1 - l \cdot \sum_{i=0}^{1-1} \frac{(-\log l)^i}{i!} \right).$$

Now suppose by way of induction that the calculation holds for $k \leq m - 1$. Then we have

$$\begin{aligned}
V_m(l) &= \int_{H(m,l)} dV = \int_{\substack{x_1 \cdots x_m \leq l \\ x_1, \dots, x_m \geq 1}} dV = \int_1^l \int_{\substack{x_1 \cdots x_{m-1} \leq l/x_m \\ x_1, \dots, x_{m-1} \geq 1}} dV dx_m \\
&= \int_1^l V_{m-1}(l/x_m) dx_m = \int_1^l (-1)^{m-1} \cdot \left(1 - (l/x_m) \cdot \sum_{i=0}^{m-2} \frac{(-\log(l/x_m))^i}{i!} \right) dx_m \\
&= (-1)^{m-1} \left(\int_1^l dx_m - l \cdot \sum_{i=0}^{m-2} \int_1^l \frac{(-\log(l/x_m))^i}{x_m \cdot i!} dx_m \right).
\end{aligned}$$

Splitting this into pieces, we see that

$$\int_1^l dx_m = l - 1,$$

and that

$$\begin{aligned}
\int_1^l \frac{(-\log(l/x_m))^i}{x_m \cdot i!} dx_m &= \int_1^l \frac{\log(x_m/l)^i}{x_m \cdot i!} dx_m = \int_{1/l}^1 \frac{(\log x)^i}{x \cdot i!} dx \\
&= \int_{-\log l}^0 \frac{u^i}{i!} du = \left(\frac{u^{i+1}}{(i+1)!} \right) \Big|_{-\log l}^0 = -\frac{(-\log l)^{i+1}}{(i+1)!}.
\end{aligned}$$

Thus,

$$\begin{aligned}
V_m(l) &= (-1)^{m-1} \left(-(1-l) - l \cdot \sum_{i=0}^{m-2} -\frac{(-\log l)^{i+1}}{(i+1)!} \right) \\
&= (-1)^m \left(1-l-l \cdot \sum_{i=1}^{m-1} \frac{(-\log l)^i}{i!} \right) = (-1)^m \left(1-l \cdot \sum_{i=0}^{m-1} \frac{(-\log l)^i}{i!} \right),
\end{aligned}$$

and we see by induction that the identity holds for all k . In particular, the recursion relation follows immediately from this identity. \square

Proof of Theorem. It is a straightforward exercise to see that $S_k(l)$ counts the number of whole unit-hypercubes contained in the solid $H(k, l)$ defined in the lemma, and so we can see that the volume $V_k(l)$ is in fact a strict upper bound for $S_k(l)$.

To prove the desired upper bound on $V_k(l)$, we apply Taylor's theorem to an inner term of the value computed in the lemma. Taking the estimate over the domain $[0, x]$, we have

$$e^{-x} = \sum_{i=0}^{k-1} \frac{(-x)^i}{i!} + R_{k-1}(x),$$

with

$$R_{k-1}(x) = \frac{(-1)^k e^{-\xi} x^k}{k!}$$

for some $\xi \in [0, x]$. In particular, $|R_{k-1}(x)| \leq |x^k/k!|$, which gives us that

$$\begin{aligned} V_k(l) &= \left| (-1)^k \left(1 - l \cdot \sum_{i=0}^{k-1} \frac{(-\log l)^i}{i!} \right) \right| \\ &= \left| 1 - l \cdot (e^{-\log l} - R_{k-1}(\log l)) \right| \\ &= |l \cdot R_{k-1}(\log l)| \leq \frac{l(\log l)^k}{k!}. \end{aligned}$$

Along with the previous observation that $S_k(l) < V_k(l)$, this proves the bound. \square

6 A Natural Extension of the Eulerian Polynomials

As introduced in [1], the Eulerian Polynomials are a sequence of polynomials $P_n(\lambda)$ original defined by Euler, and described by the relation

$$\sum_{k=0}^{\infty} (k+1)^n \lambda^k = \frac{P_n(\lambda)}{(1-\lambda)^{n+1}},$$

the first several such polynomials being

$$\begin{aligned} P_0(\lambda) &= P_1(\lambda) = 1, \\ P_2(\lambda) &= 1 + \lambda, \\ P_3(\lambda) &= 1 + 4\lambda + \lambda^2, \\ P_4(\lambda) &= 1 + 11\lambda + 11\lambda^2 + \lambda^3, \\ P_5(\lambda) &= 1 + 26\lambda + 66\lambda^2 + 26\lambda^3 + \lambda^4. \end{aligned}$$

When considering the function $1_\lambda^{-1} = (1 - \lambda \mathbf{u})^{-1}$, it is natural to notice that the Eulerian polynomials appear as the value of 1_λ^{-1} when it is evaluated at square-free numbers. That is, for distinct primes p_1, \dots, p_n ,

$$1_\lambda^{-1}(p_1 \cdots p_n) = \frac{(-1)^n P_n(\lambda)}{(1-\lambda)^{n+1}} = (-1)^n \sum_{k=0}^{\infty} (k+1)^n \lambda^k. \quad (20)$$

This natural connection between convolution inversion and power series naturally led to an investigation of analogues for square-containing numbers, which led to the following elegant identity.

Theorem 6.1. *Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a partition of a natural number, and let $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the p_i are distinct primes. Then*

$$1_\lambda^{-1}(n) = (-1)^{|\alpha|} \sum_{k=0}^{\infty} \prod_{i=1}^m \binom{k+1}{\alpha_i} \lambda^k.$$

To prove this theorem, we make use of the following lemma, adopting the notation that for $\alpha = (\alpha_1, \dots, \alpha_m)$ a partition of a natural number, $\beta \leq \alpha$ denotes that $\beta = (\beta_1, \dots, \beta_m)$ is a partition of a natural number with $\beta_i \leq \alpha_i$ for each i , and $\beta < \alpha$ denotes that $\beta \leq \alpha$ and $\beta_i < \alpha_i$ for at least one i .

Lemma 6.2. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a partition of a natural number. Then

$$-\sum_{j=0}^k \sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{j+1}{\beta_i} = (-1)^{|\alpha|} \prod_{i=1}^m \binom{k+1}{\alpha_i}.$$

Proof. We proceed by induction on k . For $k = 0$, we need to show

$$-\sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{1}{\beta_i} = (-1)^{|\alpha|} \prod_{i=1}^m \binom{1}{\alpha_i},$$

or

$$\sum_{\beta \leq \alpha} (-1)^{|\alpha|} \prod_{i=1}^m \binom{k+1}{\beta_i} = 0.$$

Assuming without loss of generality that $\alpha_i \geq 1$ for each i , we see that the non-zero summands are exactly those for which $\beta_i \in \{0, 1\}$ for each i . Summing over the number of parts β_i which are equal to 1 gives

$$\sum_{\beta \leq \alpha} (-1)^{|\alpha|} \prod_{i=1}^m \binom{1}{\beta_i} = \sum_{j=0}^m (-1)^j \binom{m}{j} = 0.$$

To prove the inductive step for $k > 0$, we make use of the identity

$$(-1)^i \binom{k}{i} = \sum_{j=0}^i (-1)^j \binom{k+1}{j}$$

to find

$$\begin{aligned} (-1)^{|\alpha|} \prod_{i=1}^m \binom{k}{\alpha_i} &= \prod_{i=1}^m (-1)^{\alpha_i} \binom{k}{\alpha_i} = \prod_{i=1}^m \sum_{j=0}^{\alpha_i} (-1)^j \binom{k+1}{j} \\ &= \sum_{\beta \leq \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{k+1}{\beta_i} = (-1)^{|\alpha|} \prod_{i=1}^m \binom{k+1}{\alpha_i} + \sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{k+1}{\beta_i}. \end{aligned}$$

Making use of the inductive hypothesis, we have

$$-\sum_{j=0}^k \sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{j+1}{\beta_i} = (-1)^{|\alpha|} \prod_{i=1}^m \binom{k}{\alpha_i} - \sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{k+1}{\beta_i} = (-1)^{|\alpha|} \prod_{i=1}^m \binom{k+1}{\alpha_i},$$

and this proves the lemma. \square

Proof of Theorem. We again proceed by induction, this time on $|\alpha|$. As a base case, suppose $|\alpha| = 0$. Then $n = 1$, and we have

$$1_{\lambda}^{-1}(1) = \frac{1}{1-\lambda} = \sum_{k=0}^{\infty} \lambda^k = (-1)^0 \sum_{k=0}^{\infty} \left(\prod_{i=1}^0 \binom{k+1}{\alpha_i} \right) \lambda^k$$

Now for the inductive step suppose that $|\alpha| > 0$. We have

$$\begin{aligned}
1_\lambda^{-1}(p_1^{\alpha_1} \cdots p_m^{\alpha_m}) &= \frac{-1}{1-\lambda} \sum_{\beta < \alpha} 1_\lambda^{-1}(p_1^{\beta_1} \cdots p_m^{\beta_m}) 1_\lambda(p_1^{\alpha_1 - \beta_1} \cdots p_m^{\alpha_m - \beta_m}) \\
&= - \sum_{\beta < \alpha} \left(\sum_{j=0}^{\infty} \lambda^j \right) \left((-1)^{|\beta|} \sum_{k=0}^{\infty} \prod_{i=1}^m \binom{k+1}{\beta_i} \lambda^k \right) = - \sum_{\beta < \alpha} \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^{|\beta|} \prod_{i=1}^m \binom{j+1}{\beta_i} \lambda^k \\
&= \sum_{k=0}^{\infty} \left(- \sum_{j=0}^k \sum_{\beta < \alpha} (-1)^{|\beta|} \prod_{i=1}^m \binom{j+1}{\beta_i} \right) \lambda^k = \sum_{k=0}^{\infty} \left((-1)^{|\alpha|} \prod_{i=1}^m \binom{k+1}{\alpha_i} \right) \lambda^k = (-1)^{|\alpha|} \sum_{k=0}^{\infty} \prod_{i=1}^m \binom{k+1}{\alpha_i} \lambda^k.
\end{aligned}$$

Here, multiplication of power series is justified by absolute convergence of both series in an open domain about the origin. By induction, we see that the theorem holds for every choice of $|\alpha|$, and so we have proven the identity in question. \square

In light of this theorem, it makes sense to extend the Eulerian polynomials to partitions of natural numbers by the relation from Equation (20), namely

$$P_\alpha(\lambda) := (1-\lambda)^{|\alpha|+1} \sum_{k=0}^{\infty} \prod_{i=1}^m \binom{k+1}{\alpha_i} \lambda^k = \left((-1)^{|\alpha|} (1-\lambda)^{|\alpha|+1} \right) 1_\lambda^{-1}(p_1^{\alpha_1} \cdots p_m^{\alpha_m}). \quad (21)$$

It remains to be verified whether known properties of the Eulerian polynomials can be naturally generalized to this class of polynomials.

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