Worksheet 21, Math H53
The Fundamental Theorem for Line Integrals

Tuesday, April 16, 2013

1. Let $\mathbf{F}(x,y) = \langle e^x \sin y, e^x \cos y\rangle$. Sketch a graph of the vector field $\mathbf{F}$, and determine whether or not $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is conservative, find a function f so that $\mathbf{F} = \nabla f$.
2. Let $\mathbf{F}(x,y) = \langle e^x \cos y, e^x \sin y\rangle$. Sketch a graph of the vector field $\mathbf{F}$, and determine whether or not $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is conservative, find a function f so that $\mathbf{F} = \nabla f$.
3. Suppose $\mathbf{F}$ is the gradient of some function. What is the work done by $\mathbf{F}$ along a *closed* curve (i.e., a curve that comes back to where it started)?
4. Show that the line integral
$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy,$$
where C is any path from $(1,0)$ to $(2,1)$, is independent of path. Evaluate the integral.
5. In the last homework, we showed that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$
Use this to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.
6. Find a function f such that $\mathbf{F}(x,y) = \langle xy^2, x^2y\rangle = \nabla f$, and use it to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve parametrized by $\mathbf{r}(t) = \langle t + \sin(\pi t/2), t + \cos(\pi t/2)\rangle$, $0 \le t \le 1$.
7. Find a function f such that $\mathbf{F}(x,y,z) = \langle yz, xz, xy + 2z\rangle = \nabla f$, and use it to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(1,0,-2)$ to $(4,6,3)$.
8. Find a function f such that $\mathbf{F}(x,y,z) = \langle \sin y, x\cos y + \cos z, -y\sin z\rangle = \nabla f$, and use it to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve parametrized by $\mathbf{r}(t) = \langle \sin t, t, 2t\rangle$, $0 \le t \le \pi/2$.
9. Suppose two different curves C and C' have the same starting point and ending point.
   (a) If $\mathbf{F}$ is the gradient of some function, must
   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r}?$$
   (b) If
   $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r},$$
   must $\mathbf{F}$ be the gradient of some function?
10. Consider
$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$
   (a) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle oriented counter-clockwise, by parametrizing C and directly integrating.
   (b) Show that $\mathbf{F}$ satisfies the condition on partial derivatives for a conservative vector field: $\partial P/\partial y = \partial Q/\partial x$.
   (c) Let D be the portion of $\mathbb{R}^2$ where $\mathbf{F}$ is defined. Is D simply connected?
   (d) Do (a) and (b) together contradict your answer to question 3?