Worksheet 14, Math H53

Lagrange Multipliers

Thursday, March 14, 2013

1. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$, subject to the constraint $xy = 1$.
2. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraint $x + y + z = 12$.
3. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^4 + y^4 + z^4$, subject to the constraint $x^2 + y^2 + z^2 = 1$.
4. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x + 2y$, subject to the constraints $x + y + z = 1$ and $y^2 + z^2 = 4$.
5. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region described by the inequality $x^2 + y^2 \leq 16$.
6. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a "piriform").
   (a) Try using Lagrange multipliers to solve the problem.
   (b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
   (c) Explain why Lagrange multipliers fail to find the minimum value in this case.
7. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral. *Hint:* Use Heron's formula for the area: $A = \sqrt{s(s-x)(s-y)(s-z)}$ where $s = p/2$ and x,y,z are lengths of the sides.
8. The base of an aquairium with given volume V is made of slate and the sides are made of glass. If slate costs $\alpha > 0$ times as much (per unit area) as glass, use Lagrange multipliers to find the dimensions of the aquarium that minimize the cost of the materials.
9. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.
10. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

   given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.
   (b) Deduce from the previous part that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

   This is the classic inequality that says that the geometric mean of n numbers is no larger than the arithmetic mean. Under what circumstances are these two means equal?