Regarding Non-homogeneous Multivariate Polynomials

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Given a quadratic polynomial $p(\mathbf{x})$, we would like to translate the coordinates by a vector $\mathbf{a}$ so that the resulting polynomial $p(\mathbf{x}-\mathbf{a})$ is homogeneous of degree 2, possibly with an additive constant. To this end, write

$$p(\mathbf{x}) = \mathbf{x}^\intercal \mathbf{A}\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c,$$

where $\mathbf{x}$ denotes the column vector of polynomial indeterminates, $\mathbf{A}$ denotes the symmetric matrix of coefficients of the quadratic terms of p (as we've seen before in the case of homogeneous quadratic polynomials), $\mathbf{b}$ is a column vector of coefficients of the linear components of p, and c is a constant. Here, the $\intercal$ symbol denotes the matrix (or vector) transpose, which switches rows with columns.

With this notation, using basic facts about matrix multiplication and the transpose operator, we have for any proposed offset vector $\mathbf{a}$ that

$$p(\mathbf{x}) = \mathbf{x}^\intercal \mathbf{A}\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c = (\mathbf{x}-\mathbf{a})^\intercal \mathbf{A}(\mathbf{x}-\mathbf{a}) + (c - \mathbf{a}^\intercal \mathbf{A}\mathbf{a}) + \mathbf{b}\cdot\mathbf{x} + \mathbf{a}^\intercal \mathbf{A}\mathbf{x} + \mathbf{x}^\intercal \mathbf{A}\mathbf{a}.$$

In particular, if we can choose $\mathbf{a}$ such that $\mathbf{b}\cdot\mathbf{x} + \mathbf{a}^\intercal \mathbf{A}\mathbf{x} + \mathbf{x}^\intercal \mathbf{A}\mathbf{a} = 0$, or equivalently that $\mathbf{b} + \mathbf{A}\mathbf{a} + \mathbf{A}^\intercal \mathbf{a} = 0$, then this choice of $\mathbf{a}$ will eliminate the non-quadratic terms in the expression. If $\mathbf{A}$ is invertible, then choosing

$$\mathbf{a} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}$$

is sufficient. If $\mathbf{A}$ is not invertible, then try to directly solve the matrix equation

$$\mathbf{A}\mathbf{a} = \mathbf{b}/2, \tag{1}$$

which is still possible for some values of $\mathbf{b}$. If it is possible to find a choice of $\mathbf{a}$ which satisfies Equation (1) the desired equation, then we have the alternate for p which is given by

$$p(\mathbf{x}) = (\mathbf{x}-\mathbf{a})^\intercal \mathbf{A}(\mathbf{x}-\mathbf{a}) + (c - \mathbf{a}^\intercal A\mathbf{a}),$$

which is a homogeneous degree 2 polynomial expression plus a constant. If there is no vector $\mathbf{a}$ which satisfies Equation (1), then there is no choice of translation which will eliminate the linear terms.