Worksheet 4, Math 54
Eigenvectors and Diagonalization

Thursday, March 13, 2014

1. True or false? Justify your answers.
   (a) If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.
   (b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
   (c) The eigenvalues of a matrix are on its main diagonal.
   (d) An eigenspace of an $n \times n$ matrix A is a null space of a certain matrix.
   (e) The determinant of an $n \times n$ matrix A is the product of the diagonal entries in A.
   (f) An elementary row operation on A does not change the determinant.
   (g) If $\lambda + 5$ is a factor of the characteristic polynomial of a matrix A, then 5 is an eigenvalue of A.
   (h) The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.
   (i) A row replacement operation on A does not change the eigenvalues.
   (j) If $\mathbb{R}^n$ has a basis of eigenvectors of a matrix A, then A is diagonalizable.
   (k) A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
   (l) If A is diagonalizable, then A is invertible.
   (m) A is diagonalizable if A has n eigenvectors.
   (n) If A is diagonalizable, then A has n distinct eigenvalues.
   (o) If $AP = PD$ for some matrices P, D, where D is diagonal, then the nonzero columns of P must be eigenvectors of A.
   (p) If A is invertible, then A is diagonalizable.

2. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

3. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

4. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A.

5. Find the characteristic polynomial of the matrix
$$A = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

6. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional:
$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

7. Show that A and A^T have the same characteristic polynomial.

8. Show that if A and B are similar, then $\det A = \det B$.
9. Diagonalize the matrices
   (a) $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$
   (b) $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$
10. Show that if A is both diagonalizable and invertible, then so is A^{-1}.
11. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.