Worksheet 2, Math 54

Matrix Algebra

Tuesday, February 11, 2014

1. True or false? Justify your answers.

   (a) For matrices A and B, each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

   (b) For matrices A, B and C, $AB + AC = A(B + C)$.

   (c) For matrices A and B, $A^T + B^T = (A + B)^T$.

   (d) If A is an $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

   (e) For matrices A, B and C, $(ABC)^T = C^T A^T B^T$.

   (f) If A and D are $n \times n$ matrices with $AD = I_n$, then $DA = I_n$.

   (g) If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then the row reduced echelon form of A is I_n.

   (h) If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution is unique.

   (i) A subset H of $\mathbb{R}^n$ is a subspace if the zero vector is in H.

   (j) If B is an echelon form of a matrix A, then the pivot columns of B form a basis for $\operatorname{Col} A$.

   (k) Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

   (l) Let H be a subspace of $\mathbb{R}^n$. If $\mathbf{x} \in H$ and $\mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x} + \mathbf{y} \in H$.

   (m) The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

   (n) Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

   (o) The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

   (p) The dimensions of $\operatorname{Col} A$ and $\operatorname{Nul} A$ add up to the number of columns in A.

   (q) If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

   (r) If $\mathcal{B}$ is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.

   (s) The dimension of $\operatorname{Nul} A$ is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.

   (t) The dimension of the column space of A is $\operatorname{rank} A$.

   (u) If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.

2. Suppose the third column of B is the sum of the first two columns. What can be said about the third column of AB? Why?

3. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

4. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

5. Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

6. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

7. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

8. Guess the inverse B of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & & 0 \\ 3 & 3 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}.$$

   Prove that your guess is correct by computing AB and BA.

9. Let

$$A = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}.$$

   Find the third column of A^{-1} without computing the other columns.

10. An $m \times n$ *upper triangular matrix* is one whose entries below the main diagonal are 0's. When is a square upper triangular matrix invertible? Justify your answer.

11. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of an $n \times n$ matrix A are linearly independent.

12. Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is B. Use this fact to show that A must also be invertible.

13. If A is an $n \times n$ matrix and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, what else can you say about this transformation? Justify your answer.

14. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?

15. Construct a 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is *not* in $\operatorname{Col} A$.

16. Suppose the columns of a matrix $A = [\mathbf{a}_1 \cdots \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is a basis for $\operatorname{Col} A$.

17. If possible, construct a 3×5 matrix A such that $\dim \operatorname{Nul} A = 3$ and $\dim \operatorname{Col} A = 2$.

18. Let A be a 6×4 matrix and B a 4×6 matrix. Show that the 6×6 matrix AB cannot be invertible.