Worksheet 1, Math 54
Vector and Matrix Equations

Tuesday, January 28, 2014

1. True or false? Justify your answers.

   (a) In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

   (b) The row reduction algorithm applies only to augmented matrices for a linear system.

   (c) A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

   (d) Finding a parametric description of the solution set of a linear system is the same as *solving* the system.

   (e) If one row in an echelon form of an augmented matrix is $\begin{bmatrix}0 & 0 & 0 & 5 & 0\end{bmatrix}$, then the associated linear system is inconsistent.

   (f) The reduced echelon form of a matrix is unique.

   (g) If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

   (h) The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

   (i) A general solution of a system is an explicit description of all solutions of the system.

   (j) Whenever a system has free variables, the solution set contains many solutions.

   (k) Another notation for the vector $\begin{bmatrix}-4\\3\end{bmatrix}$ is $\begin{bmatrix}-4 & 3\end{bmatrix}$.

   (l) The points in the plane corresponding to $\begin{bmatrix}-2\\5\end{bmatrix}$ and $\begin{bmatrix}-5\\2\end{bmatrix}$ lie on a line through the origin.

   (m) An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$.

   (n) The solution set of the linear system whose augmented matrix is $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}\end{bmatrix}$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

   (o) The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

   (p) A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   (q) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $\begin{bmatrix}A & \mathbf{b}\end{bmatrix}$ has a pivot position in every row.

   (r) The first entry in the product $A\mathbf{x}$ is a sum of products.

   (s) If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

   (t) If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then A cannot have a pivot position in every row.

2. Find the general solution of the system of equations whose augmented matrix is given by

$$\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix}$$

3. Suppose a 3×5 *coefficient* matrix for a system has three pivot columns. Is the system consistent? Why or why not?

4. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

5. Determine whether $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the vectors formed from the columns of the matrix $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

6. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

7. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

   (a) Is $\mathbf{b}$ in W?

   (b) Show that the second column of A is in W.

8. Write the following vector equation as a matrix equation:

$$x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

9. Construct a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

10. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

11. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?