Math 54 Quiz 10 Solutions
April 24, 2014

1. (a) Find the general solution to the homogeneous equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0.$$

Solution
The auxiliary equation of this ODE is

$$\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = 0,$$

which factors as $(\lambda+1)^4$ from the binomial expansion theorem. Thus the equation has a root of degree 4 at $\lambda = -1$, and the corresponding linearly independent solutions are e^{-t}, te^{-t}, t^2e^{-t} and t^3e^{-t}. The general solution is then

$$y = C_0e^{-t} + C_1te^{-t} + C_2t^2e^{-t} + C_3t^3e^{-t}.$$

(b) Represent the above equation as a system of linear equations $\mathbf{x}' = A\mathbf{x}$. What is the matrix A obtained in this representation?

Solution
To represent this equation as a system of first-order equations, we introduce variables representing different derivatives of y. If y_i represents the i-th derivative of y, then the equation can be represented using the system

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_0 - 4y_1 - 6y_2 - 4y_3 \end{cases}.$$

In matrix form, this is

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix} \mathbf{x}.$$

2. Use matrix methods to find a general solution of the first-order system

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \mathbf{x}.$$

Solution
We start by finding the eigenvalues and eigenvectors of the matrix of constant coefficients:

$$\det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & -\lambda & 3 \\ 2 & 3 & -\lambda \end{bmatrix} = -\lambda^3 + \lambda^2 + 17\lambda + 15 = 0.$$

Trying a few small integers reveals that $\lambda = -1$ is a root of this polynomial, so using long division we find the factorization

$$-\lambda^3 + \lambda^2 + 17\lambda + 15 = (\lambda+1)(\lambda^2 - 2\lambda - 15) = (\lambda+1)(\lambda+3)(\lambda-5),$$

and this gives us eigenvalues of $\lambda = -3, -1, 5$. Usual computations give us eigenvectors

$$\mathbf{v}_{-3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_{-1} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since we have 3 distinct eigenvalues, these eigenvectors are linearly independent, and we get three linearly independent solutions to the system of differential equations:

$$\mathbf{x}_1 = e^{-3t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = e^{-t} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_3 = e^{5t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the general solution is given by

$$\mathbf{x} = C_1 e^{-3t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + C_2 \mathbf{x}_2 = e^{-t} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + C_3 \mathbf{x}_3 = e^{5t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$