Math 54 Quiz 6
March 13th, 2014

1. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis $\mathcal{E} = \{1, t, t^2\}$. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

   Solution: The change-of-coordinates matrix from the basis $\mathcal{B}$ above to the standard matrix $\mathcal{E}$ is

$$P_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

   The change of coordinate matrix from the standard matrix $\mathcal{E}$ to the basis $\mathcal{B}$ is the inverse to $P_{\mathcal{B}\to\mathcal{E}}$ above, and is given by

$$P_{\mathcal{E}\to\mathcal{B}} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix}$$

   So the coordinate of t^2 with respect to the basis $\mathcal{B}$ is

$$[t^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

   Therefore, $t^2 = 3(1 - 3t^2) + (-2)(2 + t - 5t^2) + (1 + 2t)$.

2. Show that if A and B are similar, then $\det A = \det B$.

Solution: Since A and B are similar, by definition there is an invertible matrix P such that $A = PBP^{-1}$.
Then

$$det(A) = det(PBP^{-1}) = det(P)det(B)det(P^{-1}) = det(B)det(P)det(P)^{-1} = det(B).$$

3. True or False. A is an $n \times n$ matrix.

(a) If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.
False. An eigenvector needs to be nonzero.
(b) A matrix A is not invertible if and only if 0 is an eigenvalue of A.
True. This is part of the Invertible Matrix Theorem.
(c) A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = 0$ has a nontrivial solution.
True. This is how we calculate eigenvalues by solving the characteristic equation.
(d) To find the eigenvalues of A, reduce A to row echelon form.
False. Elementary row operations change the eigenvalues. Experiment with some 2×2 matrices to see this for yourself.