(a)

Math 53 Quiz 11

April 26, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Use Green's Theorem to evaluate the integral

$$\int_C x^2 y\,dx + xy^2\,dy$$

where C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

The domain enclosed by this curve is described by $0 \le x \le 1$, $x^2 \le y \le 1$, so by Green's Theorem we can express the integral as

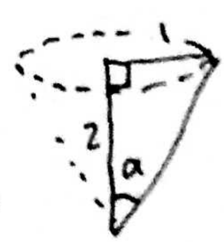
$$\int_C x^2 y\,dx + xy^2\,dy = \iint_D \frac{\partial}{\partial x} xy^2 - \frac{\partial}{\partial y} x^2 y \, dA$$

$$= \int_0^1\int_{x^2}^1 y^2 - x^2\,dy\,dx = \int_0^1 \left(\frac{y^3}{3} - yx^2\right)\Big|_{y=x^2}^{y=1} dx$$

$$= \int_0^1 (1/3 - x^2) - \left(\frac{x^6}{3} - x^4\right)dx = \left(-\frac{x^7}{21} + \frac{x^5}{5} - \frac{x^3}{3} + \frac{x}{3}\right)\Big|_0^1 = \frac{16}{105}$$

Problem 2. (3 points) Find a parametric representation for the part of the sphere $x^2 + y^2 + z^2 = 3$ that lies below the cone $z = 2\sqrt{x^2 + y^2}$.

This part of a sphere can be represented using spherical coordinates with fixed radius $\rho = \sqrt{3}$ and angle θ varying freely between 0 and 2π. The main difficulty is to find the angle formed by the cone with the z-axis.



From the diagram, we see that the angle α is given by $\tan^{-1}(1/2)$, so our final parametrization is

$$\vec{r}(\theta, \phi) = \langle \sqrt{3}\cos\theta\sin\phi, \sqrt{3}\sin\theta\sin\phi, \sqrt{3}\cos\phi \rangle,$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \tan^{-1}(1/2).$$

Problem 3. (4 points) Evaluate the surface integral

$$\iint_S z^2 + 1\, dS$$

where S is the surface with vector equation $\mathbf{r}(s,t) = \langle 2st, s^2 - t^2, s^2 + t^2 \rangle$, for $s^2 + t^2 \leq 2$.

To represent the surface integral as a standard iterated integral, we compute

$$\vec{r}_s = \langle 2t, 2s, 2s \rangle \qquad \vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2s & 2s \\ 2s & -2t & 2t \end{vmatrix} = \langle 8st, 4s^2-4t^2, -4s^2-4t^2 \rangle$$

$$\vec{r}_t = \langle 2s, -2t, 2t \rangle$$

$$|\vec{r}_s \times \vec{r}_t| = 4\sqrt{4s^2t^2 + (s^2-t^2)^2 + (s^2+t^2)^2}$$

$$= 4\sqrt{4s^2t^2 + 2s^4 + 2t^4} = 4\sqrt{2}\,|s^2+t^2| = 4\sqrt{2}(s^2+t^2).$$

Thus

$$\iint_S z^2+1\, dS = \iint_D \left((s^2+t^2)^2+1\right) \cdot 4\sqrt{2}(s^2+t^2)\, dA$$

and using polar coordinates this can be computed as

$$4\sqrt{2} \int_0^{2\pi} \int_0^{\sqrt{2}} (r^4+1)(r^2) \cdot r\, dr\, d\theta$$

$$= 8\sqrt{2}\pi \int_0^{\sqrt{2}} r^7 + r^3\, dr = 8\sqrt{2}\pi \left(\frac{r^8}{8} + \frac{r^4}{4} \right)\Big|_0^{\sqrt{2}}$$

$$= 24\sqrt{2}\,\pi.$$