Math 53 Quiz 10
April 19, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Determine whether the vector field

$$\mathbf{F}(x,y) = \left\langle 2x\ln(y-x) - \frac{x^2}{y-x}, \frac{x^2}{y-x} + 3x^2 \right\rangle$$

is conservative.

If $P(x,y) = 2x\ln(y-x) - x^2/y-x$ and $Q(x,y) = \frac{x^2}{y-x} + 3x^2$, we compute the partial derivatives:

$P_x(x,y) = 2\ln(y-x) - \frac{2x}{y-x} - \frac{2x}{y-x} - \frac{x^2}{(y-x)^2}$, $P_y(x,y) = \frac{2x}{y-x} + \frac{x^2}{(y-x)^2}$

$Q_x(x,y) = \frac{2x}{y-x} + \frac{x^2}{(y-x)^2} + 6x$, $Q_y(x,y) = \frac{-x^2}{(y-x)^2}$

The partial derivatives are continuous on the domain $y-x>0$, but $P_y \neq Q_x$, so we can conclude that $\vec{F}$ is not conservative on its domain.

Problem 2. (3 points) Find the mass of a wire following the line segment from $(0,0,0)$ to $(1,2,3)$ with density $\rho(x,y,z) = xe^{yz}$.

We can parametrize the segment by $\vec{r}(t) = \langle t, 2t, 3t \rangle$, where $0 \le t \le 1$. Then we have

$$m = \int_C \rho(x,y,z)\,ds = \int_0^1 \rho(t,2t,3t)\,|\vec{r}'(t)|\,dt$$

$$= \int_0^1 te^{6t^2} \cdot \sqrt{14}\,dt = \frac{\sqrt{14}}{12}\int_0^1 12te^{6t^2}dt \overset{\substack{u=6t^2 \\ du=12t\,dt}}{=} \frac{\sqrt{14}}{12}\int_0^6 e^u\,du$$

$$= \frac{\sqrt{14}}{12}(e^6 - 1).$$

Problem 3. (4 points) Use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = \left\langle 2xye^{x^2}, e^{x^2} + 1/y^2 \right\rangle$$

and C is the graph of the function $y = x^2 - x + 1$ for $0 \le x \le 1$.

If $\vec{F} = \nabla f$ for some function f, then we have that

$$\frac{\partial f}{\partial x} = 2xye^{x^2}, \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x^2} + 1/y^2.$$

We can find such an f by working backwards.

Since $\frac{\partial f}{\partial x} = 2xye^{x^2}$, we have that $f = \int 2xye^{x^2} dx = ye^{x^2} + g(y)$ for some function g. From this form we have that $\frac{\partial f}{\partial y} = e^{x^2} + g'(y)$, and since we already know that $\frac{\partial f}{\partial y} = e^{x^2} + 1/y^2$, we see that $g'(y) = 1/y^2$. Thus $g(y) = \int 1/y^2 \, dy = -1/y + C$, so we get a function $f(x,y) = ye^{x^2} - 1/y + C$.

To apply the fundamental theorem of line integrals, notice that the endpoints of C are given by

$$\vec{a} = (0, y(0)) = (0, 1), \quad \text{and} \quad \vec{b} = (1, y(1)) = (1, 1).$$

Thus we have

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}) = (e - 1 + C) - (1 - 1 + C) = \underline{e-1}.$$