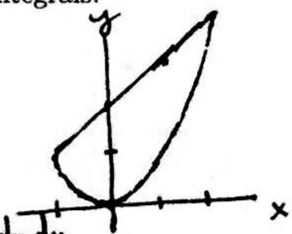
Math 53 Quiz 8
April 5, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Suppose that a lamina L occupies the region D bounded by $y = x + 2$ and $y = x^2$, and has density at each point given by $\rho(x, y) = x^2$. Write down (but don't evaluate!) expressions for the center of mass of L using iterated integrals.

The graph of D is given by:



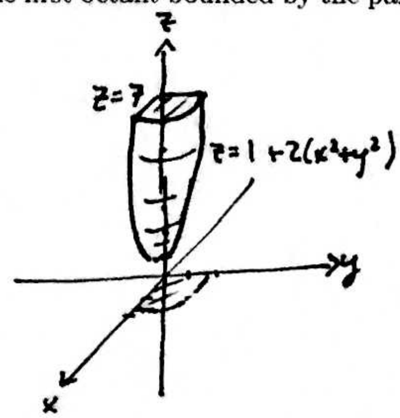
Thus integrals over D can be represented as $\iint_D f(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)\,dy\,dx$

Letting $m = \iint_D \rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} x^2\,dy\,dx$, $M_y = \iint_D x\rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} x^3\,dy\,dx$ and $M_x = \iint_D y\rho(x,y)\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} x^2 y\,dy\,dx$, the center of mass has coordinates

$$\bar{x} = M_y/m, \quad \bar{y} = M_x/m.$$

Problem 2. (3 points) Find the volume of the region in the first octant bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$.

The region can be sketched as:



In particular, the projection onto the xy plane is the quarter circle of radius $\sqrt{3}$ in the positive quadrant, and we can find the volume using polar integration:

$$V = \iint_D 7 - (1 + 2x^2 + 2y^2)\,dA = \int_0^{\pi/2}\int_0^{\sqrt{3}} (6 - 2r^2)r\,dr\,d\theta$$

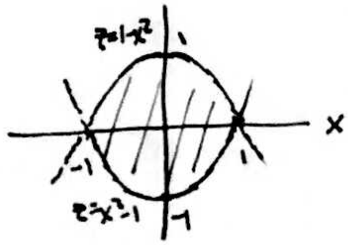
$$= \pi/2 \int_0^{\sqrt{3}} 6r - 2r^3\,dr = \pi/2\left(3r^2 - r^4/2\right)\Big|_0^{\sqrt{3}} = 9/4\,\pi.$$

Problem 3. (4 points) Evaluate the triple integral

$$\iiint_E (x-y)\,dV$$

where E is the region enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$.

E is a cylinder from $y=0$ to $y=2$ of the 2D region in the xz plane given by:



Thus the integral can be written as an iterated integral:

$$\int_{-1}^{1}\int_{x^2-1}^{1-x^2}\int_{0}^{2}(x-y)\,dy\,dz\,dx$$

$$= \int_{-1}^{1}\int_{x^2-1}^{1-x^2}\left(xy - y^2/2\right)\Big|_{y=0}^{y=2} dz\,dx = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} 2(x-1)\,dz\,dx$$

$$= \int_{-1}^{1}\left(2(x-1)z\right)\Big|_{z=x^2-1}^{z=1-x^2} dx = \int_{-1}^{1} 4(x-1)(1-x^2)\,dx$$

$$= 4\left(-x^4/4 + x^3/3 + x^2/2 - x\right)\Big|_{-1}^{1} = 4\left(2/3 - 2\right) = -16/3.$$