Math 53 Quiz 7
March 22, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find the volume of the solid enclosed by the surface $z = e^{x+y}$ and the planes $z = 0$, $x = \pm 1$, and $y = \pm 1$.

The volume is represented by the double integral

$$\iint_D e^{x+y}\,dA, \qquad D = [-1,1] \times [-1,1].$$

This can be computed as

$$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} e^x e^y\,dx\,dy = \left(\int_{-1}^{1} e^x\,dx\right)\left(\int_{-1}^{1} e^y\,dy\right)$$

$$= (e - 1/e)^2.$$

Problem 2. (3 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 1/x + 1/y$ subject to the constraint $1/x^2 + 1/y^2 = 1$.

f is not defined when x or y is zero, so assume $x, y \neq 0$. If we write the constraint as $g(x,y) = 1/x^2 + 1/y^2 = 1$, then we want to find solutions to the system of equations $\nabla f = \lambda \nabla g$, $g(x,y) = 1$. We compute $\nabla f = \langle -1/x^2, -1/y^2 \rangle$, $\nabla g = \langle -2/x^3, -2/y^3 \rangle$, so the system is:

$$\begin{cases} -1/x^2 = -2\lambda/x^3 \\ -1/y^2 = -2\lambda/y^3 \\ 1/x^2 + 1/y^2 = 1 \end{cases}$$

We can rewrite the first two equations as $x = 2\lambda$, $y = 2\lambda$, so from the last equation we get $\lambda^2 = 1/2$, or $\lambda = \pm\sqrt{2}/2$. Thus we have two possible points (x,y) which could optimize f subject to the constraint, $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Plugging into f, we find that $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ is maximal, and $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$ is minimal.

Problem 3. (4 points) Find the absolute maximum of

$$f(x, y) = x^2y^2$$

on the domain $D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 4\}$

First we check for interior extreme values by computing ∇f:

$$\nabla f = \langle 2xy^2, 2x^2y \rangle = \langle 0, 0 \rangle$$

In order for the equality to hold, it must either be the case that $x=0$ or that $y=0$. In either case, we have a value of $f(x,y)=0$.

For the boundaries of the quarter disc we have $f(x,y)=0$ along both the lines $x=0$ and $y=0$. For the quarter circle, we can parametrize by

$$\langle x, y \rangle = \langle 2\cos t, 2\sin t \rangle, \quad 0 \leq t \leq \pi/2.$$

Then we want to optimize $g(t) = f(x(t), y(t))$ on this domain. We have $g(t) = 16\cos^2(t)\sin^2(t) = 4(2\sin(t)\cos(t))^2$ $= 4\sin^2(2t)$, so $g'(t) = 16\sin(2t)\cos(2t) = 8\sin(4t)$.

We have $8\sin(4t) = 0$ when $t = 0, \pi/4,$ or $\pi/2$, giving values of $g(0)=0$, $g(\pi/4)=4$, and $g(\pi/2)=0$. Thus on the circular boundary curve, f is maximized at $(x,y)=(\sqrt{2},\sqrt{2})$, with a value of 4. This is also the global maximum.