Math 53 Quiz 6

March 15, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find the directional derivative of $f(x, y) = ye^{-x}$ at the point $(0, 4)$, in the direction represented by the angle $\theta = 2\pi/3$ on the unit circle.

The unit vector in the desired direction is given by

$$\vec{u} = \langle \cos(2\pi/3), \sin(2\pi/3) \rangle = \langle -1/2, \sqrt{3}/2 \rangle$$

The directional derivative is then computed using the gradient vector at the point in question.

$$\nabla f = \langle -ye^{-x}, e^{-x} \rangle, \quad \nabla f(0,4) = \langle -4, 1 \rangle.$$

Then

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = (-4)\cdot(-1/2) + 1\cdot(\sqrt{3}/2) = \sqrt{3}/2 + 2.$$

Problem 2. (3 points) Find and classify the local maximum and minimum points and the saddle points of $f(x, y) = e^{4y-x^2-y^2}$.

To find the critical points of f, we find the solutions of $\nabla f = 0$:

$$\nabla f = \langle -2x\, e^{4y-x^2-y^2}, (4-2y)e^{4y-x^2-y^2} \rangle = \langle 0, 0 \rangle$$

$$\rightarrow \begin{matrix} -2x = 0 \\ 4-2y = 0 \end{matrix} \rightarrow x = 0, \; y = 2.$$

Thus there's only one critical point: $(0,2)$. To classify this point, we compute the Hessian matrix:

$$H = \begin{bmatrix} (-2+4x^2)e^{4y-x^2-y^2} & -2x(4-2y)e^{4y-x^2-y^2} \\ -2x(4-2y)e^{4y-x^2-y^2} & (-2+(4-2y)^2)e^{4y-x^2-y^2} \end{bmatrix}, \quad H(0,2) = \begin{bmatrix} -2e^4 & 0 \\ 0 & -2e^4 \end{bmatrix}$$

Its determinant $D = 4e^8$ is positive, so $(0,2)$ is a max or a min. To determine which, we note that $f_{xx}(0,2) = -2e^4$ is negative, so the point is a local maximum.

Problem 3. (4 points) At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to both the lines

$$\mathbf{r}_1(t) = \langle 2, 2, 1\rangle + t\,\langle 2, -1, 0\rangle, \quad \text{and} \quad \mathbf{r}_2(t) = \langle 0, 0, 1\rangle + t\,\langle 1, 1, -1\rangle$$

The tangent plane is parallel to both lines when its normal vector is perpendicular to both of the lines' direction vectors. In other words, it should be parallel to the cross product

$$\langle 2,-1,0\rangle \times \langle 1,1,-1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1,2,3\rangle.$$

The normal vector to the tangent plane is also given by the gradient vector of the function whose level surface we are considering,

$$f(x,y,z) = x^2 - y + z^2 = 0 \longrightarrow \nabla f = \langle 2x, -1, 2z\rangle.$$

Thus we want to find a point on the surface such that

$$\langle 2x, -1, 2z\rangle = \alpha \cdot \langle 1,2,3\rangle$$

$$\begin{cases} 2x = \alpha \\ -1 = 2\alpha \\ 2z = 3\alpha \end{cases} \longrightarrow \alpha = -1/2 \longrightarrow x = -1/4,\ z = -3/4.$$

To find the point on the surface with the desired property, we compute

$$y = x^2 + z^2 = (-1/4)^2 + (-3/4)^2 = 10/16$$

Thus the desired point is $(x,y,z) = (-1/4, 10/16, -3/4)$.