Math 53 Quiz 3
February 8, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

Problem 1. (3 points) Find the acute angle between the curves $y = x^2$ and $y = x^2 - 6x + 6$ at their point of intersection. We can see that the two curves intersect at the point $x=1$, $y=1$ by solving for x in $x^2 = x^2 - 6x + 6$.

By taking derivatives, we see that the slopes of the two tangent lines are $m_1 = 2$ and $m_2 = -4$, so vectors in the directions of these lines are given by

$$\vec{v}_1 = \langle 1, m_1 \rangle = \langle 1, 2 \rangle, \quad \vec{v}_2 = \langle 1, m_2 \rangle = \langle 1, -4 \rangle.$$

Then $\cos\theta = \vec{v}_1 \cdot \vec{v}_2 / |\vec{v}_1| \cdot |\vec{v}_2| = -7/\sqrt{5}\cdot\sqrt{17}$. Since this is a negative number, $\cos^{-1}$ gives an obtuse angle, so we need to adjust by subtracting from π, giving $\theta' = \pi - \theta = \pi - \cos^{-1}\left(\frac{-7}{\sqrt{5}\sqrt{17}}\right)$.

Problem 2. (3 points) Compute the cross product $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle -5, 0, 2 \rangle$, and show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The cross product can be computed by the 3x3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -5 & 0 & 2 \end{vmatrix} = \langle 4-0, -15-2, 0+10 \rangle$$
$$= \langle 4, -17, 10 \rangle.$$

Orthogonality follows from computing dot products:

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 4 - 34 + 30 = 0$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = -20 + 0 + 20 = 0 .$$

Problem 3. (4 points) Find the equation of the plane that contains the line L_1 passing through points $A = (3, 1, 1)$ and $B = (-1, 4, 2)$, and which *does not* intersect with the line L_2 passing through points $C = (-2, -2, -1)$ and $D = (-3, -2, 0)$.

The plane that we want has a base point $\vec{x}_0 = A = \langle 3,1,1\rangle$ and two direction vectors

$$\vec{v} = \overrightarrow{AB} = \langle -1,4,2\rangle - \langle 3,1,1\rangle$$
$$= \langle -4,3,1\rangle$$
$$\vec{w} = \overrightarrow{CD} = \langle -3,-2,0\rangle - \langle -2,-2,-1\rangle$$
$$= \langle -1,0,1\rangle.$$

A normal vector is given by

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 3-0, -1+4, 0+3\rangle$$
$$= \langle 3,3,3\rangle.$$

Rescaling, we can use the normal vector $\vec{n}' = \vec{n}/3$
$= \langle 1,1,1\rangle.$

The equation of the plane is thus

$$(\vec{x}-\vec{x}_0)\cdot\vec{n}' = 0$$
$$\rightarrow (x-3)+(y-1)+(z-1) = 0$$
$$\rightarrow \underline{x+y+z=5}.$$