Math 53 Quiz 3
February 8, 2017

This quiz will be graded out of 10 points, with individual questions weighted as (indicated). No calculators, please! Please read the instructions carefully, and explain your work.

**Problem 1.** (3 points) Find two unit vectors that make an angle of 30° with the vector $\mathbf{v} = \langle -4, 3\rangle$. Compute the scalar projection of $\mathbf{v}$ onto these unit vectors. If $\vec{x}$ is a unit vector with the desired properties, then the scalar projection of $\vec{v}$ onto $\vec{x}$ is given by $\vec{x}\cdot\vec{v}/|\vec{x}| = |\vec{v}|\cos 30^\circ = 5\sqrt{3}/2$. To find $\vec{x}$, suppose it has coordinates $\langle x, y\rangle$. Then we find solutions to the equations

(1) $|\vec{x}| = 1 \longrightarrow |\vec{x}|^2 = x^2+y^2 = 1$

(2) $\vec{x}\cdot\vec{v} = |\vec{x}|\cdot|\vec{v}|\cdot\cos 30^\circ = |\vec{v}|\cdot\cos 30^\circ \longrightarrow \vec{x}\cdot\vec{v} = -4x+3y = 5\sqrt{3}/2$.

From (2), we get $y = 4/3\, x + 5\sqrt{3}/6$, and substituting into (1) gives us a quadratic formula $25x^2 + 20\sqrt{3}x + 39/4 = 0$. Solving for $x$ gives $x = \frac{-4\sqrt{3} \pm 3}{10}$, and then we can use this to find $y = \frac{3\sqrt{3} \pm 4}{10}$. The two vectors are: $\vec{x}_1 = \frac{1}{10}\langle -4\sqrt{3}+3, 3\sqrt{3}+4\rangle$, $\vec{x}_2 = \frac{1}{10}\langle -4\sqrt{3}+3, 3\sqrt{3}-4\rangle$

**Problem 2.** (3 points) Where does the line through the points $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?

We can give a parametric formula for the line using the direction vector $\vec{v} = \langle -1, 5, 6\rangle - \langle -3, 1, 0\rangle = \langle 2, 4, 6\rangle$ and base point $\vec{r}_0 = \langle -3, 1, 0\rangle$. Then the parametric formula is $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, or

$$x = -3+2t, \quad y = 1+4t, \quad z = 6t.$$

To find the point $\vec{r}(t)$ which intersects the plane, we substitute these values into the equation for the plane.

$$2x+y-z = 2(-3+2t)+(1+4t)-(6t) = -2$$

$$\longrightarrow 2t-5 = -2 \quad \longrightarrow t = 3/2.$$

The point of intersection is thus $\vec{r}(3/2) = \langle 0, 7, 9\rangle$.

**Problem 3.** (4 points) Determine whether the points $A = (1, 3, 2)$, $B = (3, -1, 6)$, $C = (5, 2, 0)$, and $D = (3, 6, -4)$ lie in the same plane in space.

One quick way to determine this is to check whether the scalar triple product of $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ is zero. We have:

$$\vec{AB} = \langle 3,-1,6\rangle - \langle 1,3,2\rangle = \langle 2,-4,4\rangle$$
$$\vec{AC} = \langle 5,2,0\rangle - \langle 1,3,2\rangle = \langle 4,-1,-2\rangle$$
$$\vec{AD} = \langle 3,6,-4\rangle - \langle 1,3,2\rangle = \langle 2,3,-6\rangle.$$

Then the scalar triple product is the determinant of the 3x3 matrix whose rows are the three vectors:

$$\begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2\cdot(-1)\cdot(-6) - 2\cdot(-2)\cdot 3 + (-4)\cdot(-2)\cdot 2 - (-4)\cdot 4\cdot(-6) + 4\cdot 4\cdot 3 - 4\cdot(-1)\cdot 2$$

$$= 12 + 12 + 16 - 96 + 48 + 8 = 0$$

Since the scalar triple product is zero, the three vectors, and thus the four points, lie in the same plane.