Worksheet 10 Solutions, Math 53

Change of Variables

Wednesday, October 31, 2012

1. Find the image of the triangular region with vertices $(0, 0)$, $(2, 2)$, and $(0, 2)$ under the transformation $x = u^2$, $y = v$.

   Solution

   The region gets mapped into a new region bounded by the line $x = 0$, the line $y = 2$, and the parabola $x = y^2$.

2. Let E be the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
   Use the change of variables $x = au$, $y = bv$ and $z = cw$ to compute the volume of E.

   Solution Idea

   The mechanics of the change of variables are straightforward, but notice that this formalizes the notion that "an ellipsoid is a stretched sphere", and gives you a rigorous way of computing the volume of the ellipsoid in terms of the volume of a sphere by using a "stretching factor".

3. Let Q be the quadrilateral in the xy-plane with vertices $(1, 0)$, $(4, 0)$, $(0, 1)$, and $(0, 4)$. Evaluate
$$\iint_Q \frac{1}{x + y} dA$$
   with the change of variables $x = u - uv$ and $y = uv$.

   Solution Idea

   One needs to find the inverse of the transformation in order to determine the new domain for the integral. Solving for u and v in terms of x and y gives us $u = x + y$ and $v = y/(x + y)$, and the new domain for the integral is given by $1 \leq u \leq 4$ and $0 \leq v \leq 1$.

4. Use a change of variables to evaluate the integral
$$\iint_R \cos\left(\frac{y - x}{y + x}\right) dA,$$
   where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, and $(0, 1)$.

   Solution Sketch

   Follow in a similar fashion to Example 3 in section 15.10, which considers the function $e^{(x+y)/(x-y)}$. In this case, the transformation to use is given by $u = y - x$ and $v = y + x$, which gives $x = (v - u)/2$ and $y = (v + u)/2$. Then the new domain of integration will be the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ in the uv-plane, and by integrating with respect to u on the inside and v on the outside, the integral is nicely solvable.

5. Find equations for a transformation from a rectangular region in the uv-plane into the parallelogram in the xy-plane with vertices $(0, 1)$, $(4, 3)$, $(2, 4)$, and $(-2, 1)$.

   Solution Sketch

   The vector from $(0, 1)$ to $(4, 3)$ is given by $\langle 4, 2\rangle$, and the vector from $(0, 1)$ to $(-2, 1)$ is given by $\langle -2, 0\rangle$, so we can use the standard linear transformation $x = 4u - 2v, y = 2u$ to map the unit square into a shifted version of the parallelogram. Finally, to map the vertex of this parallelogram at the origin to the point $(0, 1)$, we apply the additional transformation $x = u$ and $y = v + 1$. Composing these two transformations gives the final transformation of

   $$x = 4u - 2v, y = 2u + 1.$$

6. Find equations for a transformation from a rectangular region in the uv-plane into the region in the xy-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

   Solution Sketch

   Since the domain is circular, we consider a transformation from polar coordinates. Namely, on the rectangle $0 \leq \theta \leq 2\pi$ and $1 \leq r \leq 2$, we use $x = r \cos \theta$ and $y = r \sin \theta$.