Worksheet 9 Solutions, Math 53
Double and Triple Integrals

Wednesday, October 24, 2012

1. Verify that the function

$$f(x,y) = \begin{cases} 4xy & \text{if } 0 \le x \le 1,\ \ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.

If X and Y are random variables with joint density function f, find $P(X \ge \frac{1}{2})$, $P(X \ge \frac{1}{2}, Y \le \frac{1}{2})$, and the expected values of X and Y.

Solution Sketch

To verify that the function is a joint density function, we need to show that its total integral is equal to 1:

$$\int_0^1 \int_0^1 4xy\,dx\,dy = \int_0^1 4y(1^2/2 - 0^2/2)dy = \int_0^1 2y\,dy = 2(1^2/2 - 0^2/2) = 1.$$

Then the desired probabilities are given by the integrals

$$P(X \ge \frac{1}{2}) = \int_0^1 \int_{1/2}^1 4xy\,dx\,dy, \qquad P(X \ge \frac{1}{2}, Y \le \frac{1}{2}) = \int_0^{1/2} \int_{1/2}^1 4xy\,dx\,dy.$$

2. Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

Solution Idea

If we position the triangle with right angle at the origin and edges of length a along the positive x- and y-axes, then the density can be written as $\rho(x,y) = x^2 + y^2$. Then the moment M_x of the lamina about the x-axis, the moment M_y of the lamina about the y-axis, and the mass of the lamina respectively are given by

$$M_x = \iint_D y\rho(x,y)dA, \quad M_y = \iint_D x\rho(x,y)dA, \quad \text{and } m = \iint_D \rho(x,y)dA.$$

Then the center of mass is given by $(\overline{x}, \overline{y})$, where $\overline{x} = M_y/m$ and $\overline{y} = M_x/m$.

3. Find the surface area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$.

Solution Sketch

The intersection of the plane with the paraboloid is a circle of radius 5 centered on the line $(x,z) = (0,0)$, so if we think of y as a function of x and z, the part of the paraboloid we want to describe is contained in the circle $x^2 + z^2 = 25$. Then the surface area we want to find is given by

$$\iint_R \sqrt{1 + (2x)^2 + (2z)^2}dA = \int_0^{2\pi} \int_0^5 r\sqrt{1+4r^2}dr\,d\theta.$$

4. Use a triple integral to find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

   Solution Idea

   We notice that the figure is symmetric about the plane $y = 4$, so we may integrate over only half of the figure, and multiply the resulting volume by 2. Integrating with respect to y, and then with respect to a polar integral for the remaining 2 dimensions gives an integral of

$$\frac{V}{2} = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{y}} r\,dr\,d\theta\,dy.$$

5. Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

   Solution Idea

   The solid is symmetric about the xy-plane, so we may compute only the half of the volume above this plane, and then multiply this by 2. Restricting our attention to the cylinder $x^2 + y^2 = 1$ means that the r parameter should go from 0 to 1, and finding the volume inside the sphere gives us bounds on the z coordinate. Thus the integral we want is

$$\frac{V}{2} = \int_0^{2\pi} \int_0^1 \int_0^{sqrt4-r^2} r\,dz\,dr\,d\theta.$$

6. Use spherical coordinates to find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

   Solution Idea

   The necessary bounds of integration (in spherical coordinates) are fairly explicit from this problem statement. The integral we want is

$$V = \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^a \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta.$$