

Worksheet 8 Solutions, Math 53

Double Integrals

Wednesday, October 17, 2012

1. Find the average value of $f(x, y) = x^2y$ over the rectangle $R = [-1, 1] \times [0, 5]$.

Solution

The average value of a function over a domain is given by

$$\frac{\iint_R f(x, y) dA}{\iint_R 1 dA},$$

so writing the first integral as an iterated integral, we get

$$\iint_R x^2y dA = \int_{-1}^1 \int_0^5 x^2y dy dx = \frac{25}{2} \int_{-1}^1 x^2 dx = \frac{25}{3}.$$

Since the second integral is just the area of R which we can compute using geometry, we get that the average value of f over R is $(25/3)/10 = 5/6$.

2. Use symmetry to evaluate

$$\iint_R \frac{xy}{1+x^4} dA, \quad R = [-1, 1] \times [0, 1].$$

Solution Idea

Notice that the integrand is odd with respect to x , so the value of the integral over the region $R_1 = [-1, 0] \times [0, 1]$ is equal to negative of the value of the integral over the region $R_2 = [0, 1] \times [0, 1]$. Thus we have

$$\iint_R \frac{xy}{1+x^4} dA = \iint_{R_1} \frac{xy}{1+x^4} dA + \iint_{R_2} \frac{xy}{1+x^4} dA = 0.$$

3. In evaluating a double integral over a region D , a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y) dA = \int_0^1 \int_0^{2y} f(x, y) dx dy + \int_1^3 \int_0^{3-y} f(x, y) dx dy.$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

Solution

The domain of integration represented by this sum of iterated integrals is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 3)$. An alternative representation of this integral with reversed order of integration is given by

$$\iint_D f(x, y) dA = \int_0^2 \int_{x/2}^{3-x} f(x, y) dy dx.$$

4. Use geometry to evaluate the double integral

$$\iint_D \sqrt{R^2 - x^2 - y^2} dA,$$

where D is the disk with center at the origin and radius R .

Solution

The expression $\sqrt{R^2 - x^2 - y^2}$ represents the height above the point $(x, y, 0)$ of the top half of the sphere of radius R centered at the origin, so we can see that this integral is actually computing the volume of this top half of the sphere. Thus the integral evaluates to $(4\pi R^3/3)/2 = 2\pi R^3/3$.

5. Use polar coordinates to find the volume of the solid consisting of points contained inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Solution Sketch

Since we are using a double integral to find the volume of a solid, we want to represent the solid in the form $z = f(x, y)$. Since we are working with a sphere, we will have to be satisfied with representing one half of the sphere in this fashion, and then multiplying the resulting volume by 2 in order to get the volume of the whole figure. In this format, we find a function $f(x, y) = \sqrt{16 - (x^2 + y^2)}$.

The domain in the xy -plane can be represented very easily in polar coordinates by

$$R = \{(r, \theta) : 2 \leq r \leq 4, 0 \leq \theta \leq 2\pi\},$$

so if we rewrite f in terms of polar coordinates with $x = r \cos(\theta)$ and $y = r \sin(\theta)$, we get the integral

$$\int_0^{2\pi} \int_2^4 r \sqrt{16 - r^2} dr d\theta.$$

Evaluating this by substituting $u = 16 - r^2$, we find a value of $16\sqrt{3}\pi$, and so the total volume of the solid is twice this, or $32\sqrt{3}\pi$.

6. Let D be the disk with center the origin and radius a . What is the average distance from points in D to the origin?

Solution

We can rephrase this question as finding the average value of the function $f(x, y) = \sqrt{x^2 + y^2}$ over the disk D . A simple way to find this is to represent the corresponding integral in polar coordinates to get

$$\iint_D f(x, y) dA = \int_0^{2\pi} \int_0^a r^2 dr d\theta = \frac{2\pi a^3}{3}.$$

Dividing this by the area πa^2 of the disk gives an average value of $\frac{2}{3}a$.