Worksheet 2 Solutions, Math 53

Polar Coordinates

Wednesday, September 5, 2012

1. Sketch the curve given by $r = 2\sin\theta$ and give its equation in Cartesian coordinates. What curve is it?

   Solution

   The curve is a unit circle centered at the point $(0, 1)$. Notice that the polar equation sweeps out the circle twice in navigating an angle of 2π, once for the positive values of sin between 0 and π, and once for the negative values of sin between π and 2π.

2. Write an equation in polar coordinates for the circle of radius $\sqrt{2}$ centered at $(x, y) = (1, 1)$.

   Solution

   By writing out the cartesian equation for the circle

$$(x-1)^2 + (y-1)^2 = 2,$$

   and applying the cartesian to polar identities

$$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

   we can simplify to get the equation

$$r = 2(\cos(\theta) + \sin(\theta)).$$

   Notice that in this computation we have to divide through by a factor of r. However, this is okay because we don't lose any points from the solution, since we still get a point with $r = 0$ when $\theta = 3\pi/4$.

   Alternatively, we may notice that this is a circle which passes through the origin, and so we may rotate a circle centered on the x-axis by $\pi/4$ by making the substitution of $\theta - \pi/4$ for θ, giving us the equation

$$r = \sqrt{2}\cos(\theta - \pi/4).$$

3. (a) Does the spiral $r = 1/\theta, \pi/2 \leq \theta < \infty$ have finite length?

   Solution

   The length of the spiral swept out by the equation for $\pi/2 \leq \theta < a$ is given by the equation

$$L_a = \int_{\pi/2}^{a} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\pi/2}^{a} \sqrt{\frac{\theta^2 + 1}{\theta^4}}\, d\theta.$$

   Taking the limit as $a \to \infty$ of this length will give us the length of the spiral:

$$L = \lim_{a\to\infty} L_a = \int_{\pi/2}^{\infty} \sqrt{\frac{\theta^2+1}{\theta^4}}\, d\theta.$$

   So we need only consider the question of whether this improper integral converges. But we have in particular that

$$\sqrt{\frac{\theta^2+1}{\theta^4}} > \frac{1}{\theta}$$

for every positive θ, so because the integral

$$\int_{\pi/2}^{\infty} \frac{1}{\theta} d\theta$$

is divergent, this implies that the integral in question is also divergent by the comparison test. Thus we can tell that this spiral has infinite length.

(b) Does the spiral $r = e^{-\theta}, 0 \leq \theta < \infty$ have finite length?

Solution
Similarly to the above question, we write

$$L = \int_0^{\infty} \sqrt{2} e^{-\theta} d\theta,$$

but in this case, we can easily see that the improper integral is finite, and so we conclude that this spiral has finite length.

4. Find the points on the spiral $r = e^{\theta}$ where the tangent line is horizontal or vertical.

Solution
We compute $dx/d\theta$ and $dy/d\theta$, and find points where one but not both are zero. Using

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos\theta - r \sin\theta, \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin\theta + r\cos\theta$$

we find

$$\frac{dx}{d\theta} = e^{\theta}(\cos\theta - \sin\theta),$$

and

$$\frac{dy}{d\theta} = e^{\theta}(\sin\theta + \cos\theta).$$

Thus $dx/d\theta$ is zero when $\cos\theta - \sin\theta = 0$, or $\theta = \pi/4 + \pi k$ for $k \in \mathbb{Z}$, and $dy/d\theta$ is $\sin\theta + \cos\theta = 0$, or $\theta = 3\pi/4 + \pi k$ for $k \in \mathbb{Z}$. Since there is no place where both of these derivatives are zero, each zero of $dx/d\theta$ is a vertical tangent, and each zero of $dy/d\theta$ is a horizontal tangent.

5. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

Solution Idea
Drawing a picture shows that the larger loop is traced out for θ between $-2\pi/3$ and $2\pi/3$, and the smaller loop is traced out for θ between $2\pi/3$ and $4\pi/3$. Thus the total area will be determined by computing the area of the larger loop, and subtracting off the area of the smaller loop, using the proper integral formula for the area swept out by a polar equation, over their corresponding ranges of θ.