Worksheet 1 Solutions, Math 53

Parametric Equations

Wednesday, August 29, 2012

1. For the following parametric equations, sketch the curve represented by the equations, indicating with an arrow the direction in which the curve is traced as the parameter increases, and eliminate the parameter to find a Cartesian equation of the curve.

   (a) $x = 1 - t^2$, $y = t - 2$, $-2 \leq t \leq 2$

   (b) $x = \sin t$, $y = \csc t$, $0 < t < \pi/2$

   (c) $x = e^{2t}$, $y = t + 1$

   (d) $x = \tan^2 \theta$, $y = \sec \theta$, $-\pi/2 < \theta < \pi/2$

2. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[0, 4]$ and the range of g is $[-2, 1]$. What can you say about the curve?

   Solution

   In this case, we can say that the graph of the curve is contained in the box with x values between 0 and 4, and y values between -2 and 1.

3. Compare the curves represented by the following parametric equations. How do they differ?

   (a) $x = t^3$, $y = t^2$

   (b) $x = t^6$, $y = t^4$

   (c) $x = e^{-3t}$, $y = e^{-2t}$

   Solution

   Eliminating parameters shows us that the graphs of the functions all look mostly alike, as some subset of the graph of $y = \sqrt[3]{x^2}$. However, from the previous problem, we see that the second and third equations give graphs which are restricted to the first quadrant, while the first equation gives a graph which has negative x values. Further considering the asymptotic behaviors of the latter two equations reveals that the first tends to infinity for both large positive t and large negative t, while the second one only tends to infinity for large negative t, and instead tends toward the origin for large positive t.

4. Let P be a point at a distance d from the center of a circle of radius r, and consider the *trochoid* traced out by P as the circle rolls along a straight line. Assuming that

   - The circle starts out sitting on the origin,
   - The circle rolls in the negative direction along the x-axis,
   - The circle rolls at a rate of θ radians per unit time, and
   - The point P starts out directly above the center of the circle,

   find the parametric equations of P in terms of the time parameter t.

   Solution Idea

   This is roughly the same construction as that of finding the parametric equations of the cycloid. Instead of following a point along the wheel as it rolls, the point will be situated along a "spoke" of the wheel. The resulting equation is

$$x(t) = -r\theta t - d\sin(\theta t), \qquad y(t) = r + d\cos(\theta t).$$