Math 480A2, Homework 2

Due September 8, 2022

Homework is graded out of a total of 10 points. Collaboration is permitted, but you must list all coauthors on a problem's solution at the top of the page, and your writing must be your own.

Problem 1. (3 points) Describe the elements which are units in the ring $R = \mathbb{Z}/12\mathbb{Z}$. List the ideals of R (express each as a set), and explain why each set is an ideal, and why every ideal is accounted for.

Problem 2. (2 points) Let R, R' be commutative rings, let $\varphi : R \to R'$ be a homomorphism, and let I' be an ideal in R'. Prove that the preimage

$$\varphi^{-1}(I') = \{r \in R : \varphi(r) \in I'\}$$

is an ideal in R.

Problem 3. (2 points) Let $f(x) = 3x^5 + x^4 + 5x^3 + 4x^2 + 4x + 6$ and let $g(x) = x^3 + x + 1$ be polynomials in $\mathbb{Z}[x]$. Use polynomial long division to find quotient and remainder polynomials q and r such that $f(x) = q(x)g(x) + r(x)$, where r has degree less than the degree of g.

Problem 4. (3 points) Let $R = \mathbb{Z}/2\mathbb{Z}$, and let $F = R[x]/(x^2+x+1)$, where x^2+x+1 is interpreted as a polynomial with coefficients in $\mathbb{Z}/2\mathbb{Z}$. How many elements does F have? (Hint: use polynomial long division to express each element of F in a simple "canonical" form.) Write out the addition and multiplication table of F, and explain why F is a field.