Worksheet 7 Solutions, Math 1B
Power Series

Monday, March 5, 2012

1. Find the radius of convergence and interval of convergence of the series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   Solution Sketch

   Ratio test gives a radius of convergence of $R = 1$. Testing the endpoints of $(-1, 1)$ gives convergence at $x = -1$ using the alternating series test, and divergence at $x = 1$ as a p-series with $p \leq 1$. Thus the interval of convergence is $[-1, 1)$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

   Solution Sketch

   Ratio test gives a radius of convergence of $R = 5$. Testing the endpoints of $(-5, 5)$ gives convergence at $x = -5$ using the alternating series test, and convergence at $x = 5$ as a p-series with $p > 1$. Thus the interval of convergence is $[-5, 5]$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

   Solution Sketch

   Ratio test gives a radius of convergence of $R = 4$ (about the point $a = -1$). Testing the endpoints of $(-5, 3)$ gives divergence at both $x = -5$ and $x = 3$ by the divergence test. Thus the interval of convergence is $(-5, 3)$.

2. Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

   Solution

   If we let $f(x) = \sum c_n x^n$, then the second series is just $f(x^2)$. Suppose first that R is finite and nonzero. Since $f(x)$ converges for $|x| < R$ and diverges for $|x| > R$, we see that $f(x^2)$ converges for $\left|x^2\right| < R$ and diverges $\left|x^2\right| > R$. Thus we see that the radius of convergence of the new series is just $\sqrt{R}$.

   If R is equal to positive infinity, then $f(x^2)$ converges for every possible input, and so the radius of convergence of the new series is also positive infinity.

   If R is equal to 0, then $f(x^2)$ diverges for every value of x not equal to 0, and so the radius of convergence of the new series is also 0.

3. Find a power series representation for the function and determine the radius of convergence:

   (a) $f(x) = \dfrac{1+x}{1-x}$

   Solution
   We write

$$\begin{aligned} f(x) &= \frac{1}{1-x} + x \cdot \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n + x \cdot \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} x^n \\ &= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots \end{aligned}$$

   Applying the ratio test to the resulting series gives a radius of convergence of 1.

   (b) $f(x) = \ln(5 - x)$

   Solution Sketch
   We can differentiate to find $f'(x) = -1/(5-x) = -1/5 \cdot 1/(1-(x/5))$. This we can transform to a power series with radius of convergence 5 using the geometric series, and integrating (term by term) gives a series for the original function, also with radius of convergence 5. Compute the correct value for the constant of integration by comparing with our target function f.

   (c) $f(x) = \dfrac{x^3}{(x-2)^2}$

   Solution Sketch
   We can rewrite f as

$$f(x) = x^3 \cdot \frac{1}{(2-x)^2},$$

   and we notice that

$$\frac{1}{(2-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2-x}\right),$$

   for which we can find a power series with radius of convergence 2 by using the geometric series. Differentiating and substituting back into the formula for f gives a series representation which also has radius of convergence 2.

4. Find the sums of the following series:

   (a) $\displaystyle\sum_{n=1}^{\infty} n x^{n-1}, \quad |x| < 1$

   Solution Sketch
   This is the (term by term) derivative of the geometric series, so we see that it has a value equal to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

(b) $\sum_{n=1}^{\infty} nx^n, \quad |x| < 1$

Solution
Making use of the last problem, we see that this series is just

$$x \cdot \sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

(c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Solution
This value we can obtain by substituting 1/2 into our last solution, giving

$$\frac{(1/2)}{(1-(1/2))^2} = 2.$$

(d) $\sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$

Solution Sketch
Differentiating the series from the first part a second time gives

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

Multiplying by x^2 gives the value of the series in question:

$$\frac{2x^2}{(1-x)^3}.$$

(e) $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$

Solution
Again substituting 1/2 into the result of the previous problem gives a value for the sum:

$$\frac{2(1/2)^2}{(1-(1/2))^3} = 4.$$

(f) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Solution Sketch
We can obtain the general series (in terms of x, where this is evaluated at $x = 1/2$) by differentiating the series

$$\sum_{n=1}^{\infty} nx^n,$$

and multiplying by x. This gives a value for the general sum of

$$x \cdot \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{x(1+x)}{(1-x)^3},$$

and substituting $x = 1/2$ gives a final sum of

$$\frac{(1/2)(1+(1/2))}{(1-(1/2))^3} = 6.$$

5. Fix an integer $k > 0$, and let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+k} = c_n$ for all $n \geq 0$. Assume that f is not a constant function. Find the interval of convergence of the series, and a formula for $f(x)$.

Solution

The condition that $c_{n+k} = c_n$ for all n implies that there are only k possible values for the coefficients of the series, and we can name them

$$\begin{aligned} a_0 &= c_0 = c_k = c_{2k} = \cdots \\ a_1 &= c_1 = c_{k+1} = c_{2k+1} = \cdots \\ &\vdots \\ a_{k-1} &= c_{k-1} = c_{2k-1} = c_{3k-1} = \cdots \end{aligned}$$

The case where $a_i = 0$ for every i is special, because we can easily see that in this case the series converges to $f(x) = 0$ for all values of x, and the interval of convergence is $(-\infty, \infty)$.

So assume that at least one of the coefficients is non-zero. In this case, let $M = \max\{|a_1|, |a_2|, \ldots, |a_k|\} > 0$. We notice that

$$|c_n x^n| \leq M\,|x|^n\,,$$

and so by the comparison test we see that our series is absolutely convergent for $|x| < 1$, hence convergent. However, for $x = \pm 1$, we see that the terms of our series do not go to 0, and so the series diverges at these points. In particular, this means that the radius of convergence cannot be larger than 1, since that would imply convergence at $x = \pm 1$, so we see that the radius of convergence is 1, and the interval of convergence is $(-1, 1)$.

In particular, for x in $(-1, 1)$, the series is absolutely convergent, and this means that we can rearrange the terms into a more convenient order while preserving convergence to the same value:

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^n &= \sum_{i=0}^{k-1} \sum_{m=0}^{\infty} c_{mk+i} x^{mk+i} = \sum_{i=0}^{k-1} \sum_{m=0}^{\infty} a_i x^{mk+i} = \sum_{i=0}^{k-1} a_i x^i \sum_{m=0}^{\infty} (x^k)^m \\ &= \sum_{i=0}^{k-1} a_i x^i \frac{1}{1-x^k} = \frac{\sum_{i=0}^{k-1} a_i x^i}{1-x^k} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1}}{1-x^k} \end{aligned}$$

Thus by separating terms of the series which have the same coefficient, we can factor out a geometric series with ratio x^k in order to obtain a simple closed form for the series.