

# Worksheet 4 Solutions, Math 1B

## Numerical Integration, Improper Integrals

Friday, February 3, 2012

1. Consider the integral  $\int_1^2 e^{1/x} dx$ . How large do we have to choose  $n$  so that, respectively, the trapezoid and midpoint approximations with  $n$  subintervals are accurate within an error bound of  $10^{-4}$ ?

*Solution Idea*

We calculate for  $f(x) = e^{1/x}$  that

$$f''(x) = e^{1/x} (2x^{-3} + x^{-4}),$$

and we bound the magnitude of its value above by

$$\begin{aligned} |f''(x)| &= \left| e^{1/x} (2x^{-3} + x^{-4}) \right| = \left| e^{1/x} \right| |2x^{-3} + x^{-4}| \\ &\leq \left| e^{1/x} \right| (|2x^{-3}| + |x^{-4}|) = \left| e^{1/x} \right| (2|x^{-3}| + |x^{-4}|) \leq e(2 \cdot 1 + 1) = 3e. \end{aligned}$$

So  $K = 3e$  will serve in the error bound equations for the midpoint and trapezoid error bounds. We need only calculate  $n$  such that the error bound expressions yield a value of less than  $10^{-4}$ .

2. Sketch the graph of a continuous function on  $[0, 2]$  for which the right endpoint approximation with  $n = 2$  is more accurate than Simpson's Rule.

*Solution Idea*

Writing out the corresponding approximations, we have

$$R_2 = f(1) + f(2), \quad S_2 = f(0)/3 + 4f(1)/3 + f(2)/3.$$

In particular, we see that the right endpoint approximation does not depend on the value of  $f$  at 0, while the Simpson's approximation does. This suggests the following approach: Let  $f(0)$  be very large, but then have  $f$  decrease quickly to 0 (and remain there) as  $x$  increases from 0.

3. Find the *escape velocity*  $v_0$  that is needed to propel a rocket of mass  $m$  out of the gravitational field of a planet with mass  $M$  and radius  $R$ . Use Newton's Law of Gravitation

$$F = G \frac{m_1 m_2}{r^2},$$

and the fact that the initial kinetic energy of  $\frac{1}{2}mv_0^2$  supplies the needed work.

*Solution Idea*

The energy needed to escape the gravitational field, i.e. to start moving in such a way that you will never be pulled back to the planet regardless how long you travel, is equal to the work done by the force of gravity as the rocket moves out to infinity from the planet's surface. If we parametrize the rocket's position by its distance  $r$  from the center of the planet, this gives us an energy (equal to the work) of

$$E = W = \int_R^\infty F(r) dr = \int_R^\infty G \frac{mM}{r^2} dr,$$

and once we calculate this integral, we need only set the result equal to the initial kinetic energy and solve for  $v_0$  in order to find the escape velocity.

4. Show that if  $a > -1$  and  $b > a + 1$ , then the following integral is convergent:

$$\int_0^{\infty} \frac{x^a}{1+x^b} dx.$$

*Solution*

Notice that the integral above can be considered as both a type 1 and a type 2 integral, since the domain of integration goes to positive infinity, and the function may have (for  $a < 0$ ) a discontinuity at 0. Thus to show that the overall integral is convergent, we need to split it up into two separate intervals, and show convergence of each of

$$\int_0^1 \frac{x^a}{1+x^b} dx \quad \text{and} \quad \int_1^{\infty} \frac{x^a}{1+x^b} dx.$$

(The boundary point of 1 is only chosen for convenience, and other choices would work as well.) We have seen in class that  $\int_1^{\infty} x^p$  is convergent when  $p < -1$ , and divergent when  $p \geq -1$ . We can similarly show by direct calculation that  $\int_0^1 x^p$  is convergent when  $p > -1$ , and divergent when  $p \leq -1$ . These two important facts will play a crucial role in proving our result here.

To bound  $x^a/(1+x^b)$  above, we rewrite it as

$$\frac{x^a}{1+x^b} = \frac{1}{x^{-a} + x^{b-a}},$$

and notice that since we are working with only non-negative values of  $x$ , each of the two terms in the denominator is also non-negative. Thus

$$\frac{1}{x^{-a} + x^{b-a}} \leq \frac{1}{x^{b-a}} = x^{a-b} \quad \text{and} \quad \frac{1}{x^{-a} + x^{b-a}} \leq \frac{1}{x^{-a}} = x^a,$$

and so both  $x^a$  and  $x^{a-b}$  are upper bounds for  $x^a/(1+x^b)$  for non-negative  $x$ . In particular, the conditions on the problem require that  $a-b < -1$ , and so the bound of  $x^{a-b}$  allows us to use the comparison theorem to show that the integral on 1 to  $\infty$  is convergent. Likewise, the condition that  $a > -1$  with the bound of  $x^a$  allows us to use the comparison theorem to show that the integral on 0 to 1 is also convergent.

Thus by splitting the integral and applying separate bounds to the parts, we have shown that both parts are convergent, and so we see that the overall integral from 0 to  $\infty$  converges as well.

5. Show that if  $f$  is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of  $\int_a^b f(x) dx$ .

*Solution Idea*

Because both integrals and the integral approximations are linear (split sums and allow you to take out constant multiples), it's sufficient to show that a single step Simpson's approximation

$$\int_a^{a+2\Delta x} f(x) \approx \frac{\Delta x}{3} (f(a) + 4f(a + \Delta x) + f(a + 2\Delta x))$$

calculates the integral precisely for  $f(x) = 1$ ,  $f(x) = x$ ,  $f(x) = x^2$ , and  $f(x) = x^3$ .

6. Find the value of the constant  $C$  for which the integral

$$\int_0^{\infty} \left( \frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx$$

converges. Evaluate the integral for this value of  $C$ .

*Solution Sketch*

The idea is to first combine the two fractions, and then to see how the choice of constant  $C$  affects the structure of the result. Indeed, we see that

$$\frac{x}{x^2 + 1} - \frac{C}{3x + 1} = \frac{(3 - C)x^2 + x - C}{3x^3 + x^2 + 3x + 1},$$

and so it is reasonable to guess that  $C = 3$  would be a good choice since it knocks out the  $x^2$  term in the numerator, meaning that the result has a difference of two degrees between the polynomial in the numerator and the polynomial in the denominator. With this choice of  $C$ , we can use partial fractions and the limit definition of a type 1 improper integral to solve.

Although it's not necessary to show for this problem, it is also possible to prove that the integral diverges for  $C \neq 3$  by using the comparison theorem.