Worksheet 2 Solutions, Math 1B

Trigonometric Substitution

Monday, January 23, 2012

1. Evaluate the integral

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx$$

using trigonometric substitution. Then evaluate it using integration by parts.

Solution Idea

Trigonometric substitution calls for a tangent-type substitution. Integration by parts makes use of the initial factorization $u = x^2$ and $dv = x/\sqrt{4+x^2}dx$.

2. Evaluate the following integrals:

(a) $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx$

Solution

We use a tangent-type substitution, $x = 2\tan(\theta)$ (where $-\pi/2 < \theta < \pi/2$), which gives us $dx = 2\sec^2(\theta)d\theta$, and

$$\begin{aligned}\int_0^1 x\sqrt{x^2+4}\,dx &= \int_{x=0}^{x=1} (2\tan(\theta))\sqrt{(2\tan(\theta))^2+4}\,(2\sec^2(\theta))\,d\theta \\ &= 8\int_{x=0}^{x=1} \tan(\theta)\sqrt{\tan^2(\theta)+1}\,\sec^2(\theta)\,d\theta \\ &= 8\int_{x=0}^{x=1} \tan(\theta)\sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = 8\int_{x=0}^{x=1} \tan(\theta)\sec^3(\theta)\,d\theta\end{aligned}$$

From this we can use a simple substitution $u = \sec(\theta)$ to find a value of

$$(8\sec^3(\theta)/3)\Big|_{x=0}^{x=1}.$$

Using a right triangle to represent the substitution $x = 2\tan(\theta)$, we find that $\sec(\theta) = \sqrt{x^2+4}/2$, and so the final result is

$$\Big((x^2+4)^{3/2}/3\Big)\Big|_{x=0}^{x=1} = (\sqrt{125}-8)/3.$$

Alternatively, a simpler solution uses substitution with $u = x^2+4$.

(b) $\int \sqrt{5+4x-x^2}\,dx$

Solution Idea

Completing the square yields

$$\int \sqrt{5+4x-x^2}\,dx = \int \sqrt{9-(x-2)^2}\,dx,$$

which calls for a sine-type substitution after first substituting $u = x-2$.

(c) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

Solution Sketch

A first quick substitution of $u = ax$ gives

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = a^4\int_0^1 u^2\sqrt{1-u^2}\,du,$$

and from here a sine-type substitution gives us an integral of the form

$$\int \sin^2(\theta)\cos^2(\theta)\,d\theta.$$

Several applications of the half-angle formulas allow a solution.

(d) $\int \frac{x^2}{9-25x^2}\,dx$

Solution Idea

A substitution of $x = (3/5)\sin(\theta)$ makes this tractable.

(e) $\int \frac{1-\tan^2 x}{\sec^2 x}\,dx$

Solution Idea

Converting into sines and cosines and simplifying shows that this integral is actually equal to

$$\int \cos(2x)\,dx.$$

(f) $\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

Solution Idea

Make an initial substitution of $u = \sin(t)$ to turn this into a usual trig substitution integral, which is amenable to a tangent-type substitution.

3. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

Solution Idea

The cross-section of the circle at a given value of $x \in [-r, r]$ goes from $R - \sqrt{r^2 - x^2}$ to $R + \sqrt{r^2 - x^2}$, and so the cross-sectional area of the solid of revolution is given by

$$\pi(R + \sqrt{r^2 - x^2})^2 - \pi(R - \sqrt{r^2 - x^2})^2.$$

Then the overall volume is an integral over all the values of x of the cross-sectional area, which is

$$V = \int_{-r}^{r} \pi(R + \sqrt{r^2 - x^2})^2 - \pi(R - \sqrt{r^2 - x^2})^2 = 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2}.$$

This can be solved using a simple sine-style trig substitution, but alternatively we can see that the integral is equal to the area above the x-axis of the circle of radius r centered at the origin, which is $(\pi r^2)/2$. This gives us a final volume of $2\pi R r^2$.