Worksheet 13 Solutions, Math 1B
Damped Harmonic Motion and Series Solutions

Friday, April 20, 2012

1. A spring with a mass of 2 kg has a damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t. Determine also what mass of weight would result in critical damping in this system.

   Solution

   The general differential equation describing damped motion is of the form

   $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t),$$

   where m is the mass of the weight, c is the damping constant, k is the spring constant, and $F(t)$ is the external force acting on the spring. Here we are given the mass, the damping constant, and the external force, and we need to calculate the spring constant based on the fact that a force of 6 N is required to keep the spring stretched 0.5m beyond its natural length. Since the restoring force is equal to k times the displacement on an ideal spring by Hooke's Law, we conclude that the spring constant is $6/0.5 = 12$.

   Thus we have $m = 2$, $c = 14$, $k = 12$, and $F(t) = 0$, and the differential equation is given by

   $$2x'' + 14x' + 12x = 0.$$

   The characteristic equation is given by

   $$2r^2 + 14r + 12 = 0,$$

   and so the roots are $r = -6, -1$, giving a general solution of

   $$x = C_1e^{-6t} + C_2e^{-t}.$$

   The initial conditions of the problem give us $x(0) = 1$ and $x'(0) = 0$, and the derivative of x is $x' = -6C_1e^{-6t} - C_2e^{-t}$, so plugging in the initial conditions, we have

   $$\begin{cases} C_1 + C_2 = 1 \\ -6C_1 - C_2 = 0 \end{cases}$$

   From this, we conclude $C_1 = -1/5, C_2 = 6/5$, and so the solution to this initial value problem is

   $$x = \frac{6}{5}e^{-6t} - \frac{1}{5}e^{-t}.$$

   Critical damping occurs when the descriminant $c^2 - 4mk$ is zero, so that means we would require a weight which has mass $m = c^2/(4k) = 49/12$.

2. Use power series to solve the initial value problem. For each series, determine the interval of convergence.

(a) $y'' - y = 1, \quad y(0) = 0, \quad y'(0) = 0$

Solution Sketch
Supposing that $y = \sum a_n x^n$, we plug this general form into the differential equation to find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 1,$$

and reindexing gives us

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - a_n\right) x^n = 1.$$

This determines the following relations on the coefficients:

$$\begin{cases} a_2 = \frac{a_0+1}{2} \\ a_{n+2} = \frac{a_n}{(n+2)(n+1)}, \quad n \geq 1 \end{cases}.$$

From these relations we deduce the following pattern of the coefficients:

$$\begin{cases} a_{2k} = \frac{a_0+1}{(2k)!}, & k \geq 1 \\ a_{2k+1} = \frac{a_1}{(2k+1)!}, & k \geq 0 \end{cases}$$

Then splitting up y so that the sum is over terms which have the same class of coefficients, we see that

$$y = \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} = a_0 + (a_0+1)\sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Applying the initial conditions to y then gives us that $a_0 = 0$ and $a_1 = 0$, so the solution of the initial value problem is

$$y = \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!}.$$

(b) $y'' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

Solution Sketch
Supposing that $y = \sum a_n x^n$, we plug this general form into the differential equation to find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

and reindexing gives us

$$2a_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} + a_{n-1}\right) x^n = 0.$$

This determines the following relations on the coefficients:

$$\begin{cases} a_2 = 0 \\ a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1 \end{cases}.$$

From these relations we deduce the following pattern of the coefficients:

$$\begin{cases} a_{3k} = \frac{(-1)^k a_0}{(3k)(3k-1)(3k-3)(3k-4)\cdots(3)(2)}, & k \geq 1 \\ a_{3k+1} = \frac{(-1)^k a_1}{(3k+1)(3k)(3k-2)(3k-3)\cdots(4)(3)}, & k \geq 1 \\ a_{3k+2} = 0, & k \geq 0 \end{cases}$$

Then splitting up y so that the sum is over terms which have the same class of coefficients, we see that

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_{3k}x^{3k} + \sum_{k=0}^{\infty} a_{3k+1}x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2}x^{3k+2} \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{(3k)(3k-1)(3k-3)(3k-4)\cdots(3)(2)} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{(3k+1)(3k)(3k-2)(3k-3)\cdots(4)(3)} \end{aligned}$$

Applying the initial conditions to y then gives us that $a_0 = 0$ and $a_1 = 1$, so the solution of the initial value problem is

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{(3k+1)(3k)(3k-2)(3k-3)\cdots(4)(3)}.$$

(c) $y'' + \frac{2y'}{x} = 0, \quad y(1) = 1, \quad y'(1) = -1$

Solution Idea

Use similar techniques as in the above problems, with a couple of small modifications. First, make used of a different base point for your power series solution in order to match the base point $x_0 = 1$ of the initial conditions:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Second, rewrite the differential equation in the form

$$y'' + (x-1)y'' + 2y' = 0$$

to more easily simplify after substituting the power series solution.

(d) $x^2y'' + xy' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

Solution Idea

Use similar techniques as in the above problems.

See `http://en.wikipedia.org/wiki/Bessel_function` for a more detailed introduction to Bessel functions and their applications.

The solution to the last of these initial value problems is called a Bessel function of order 0.