Worksheet 12 Solutions, Math 1B
Complex Numbers and Nonhomogeneous Linear Equations

Monday, April 16, 2012

1. Solve the differential equation or initial value problem using the method of undertermined coefficients.

   (a) $y'' + 3y' + 2y = x^2$

   Solution
   We first find a solution to the homogeneous differential equation. the characteristic equation is

   $$r^2 + 3r + 2 = 0,$$

   which has roots $r = -1, -2$ corresponding to solutions

   $$y_1 = e^{-t}, \quad y_2 = e^{-2t}.$$

   We propose the particular solution $y_p = Ax^2 + Bx + C$, and we note that none of the linearly independent parts x^2, x, and 1 are solutions to the homogeneous equation. Thus this proposed solution should be sufficient. We substitute the general proposed solution into the nonhomogeneous equation to get

   $$\begin{aligned}&2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C\\ &= (2A)x^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2.\end{aligned}$$

   Equating the coefficients of corresponding linearly independent functions gives us the system of equations

   $$\begin{cases} 2A & = 1 \\ 6A + 2B & = 0 \\ 2A + 3B + 2C & = 0 \end{cases},$$

   which has solution $A = 1/2$, $B = -3/2$, and $C = 7/4$. Thus a particular solution to the differential equation is

   $$y_p = x^2/2 - 3x/2 + 7/4,$$

   and the general solution is

   $$y = (x^2/2 - 3x/2 + 7/4) + C_1 e^{-t} + C_2 e^{-2t}.$$

   (b) $y'' + 9y = e^{3x}$

   Solution Idea
   The homogeneous equation has solutions

   $$y_1 = \cos(3x), \quad y_2 = \sin(3x),$$

   and so the proposed form of the solution is

   $$y_p = Ae^{3x}.$$

(c) $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$

Solution Idea

The homogeneous equation has solutions

$$y_1 = e^{2x}, \quad y_2 = e^{-2x},$$

so the proposed form of the solution is

$$y_p = Ae^x \cos x + Be^x \sin x.$$

2. Find all solutions to the equation $x^4 = 1$.

Solution

Writing a complex number in polar form as $z = re^{i\theta}$, the equation is equivalent to

$$r^4 e^{i(4\theta)} = 1.$$

Since both sides must have magnitude 1, we conclude that $r = 1$, and so we only need to find the values of θ such that $e^{i(4\theta)} = 1$. Using Euler's formula, this gives us

$$\cos(4\theta) + i\sin(4\theta) = 1,$$

and so we see that 4θ must be a multiple of 2π, or $\theta = k\pi/2$ for any integer k. But the values of z corresponding to these values of θ repeat every 4 values, so we see that the only solutions to the equation are

$$\begin{cases} z = e^{i(0)} & = 0 \\ z = e^{i(\pi/2)} & = i \\ z = e^{i(\pi)} & = -1 \\ z = e^{i(3\pi/2)} & = -i \end{cases}.$$

3. Write a trial solution for the method of undertermined coefficients for the differential equation

$$y'' + 3y' - 4y = (x^3 + x)e^x$$

Do not determine the coefficients.

Solution Sketch

The solutions to the homogeneous equation are

$$y_1 = e^x, \quad y_2 = e^{-4x}.$$

We propose the form

$$(Ax^3 + Bx^2 + Cx + D)e^x$$

for the trial solution, but notice that e^x is a solution to the homogeneous equation. Thus we multiply this form by x and propose

$$y_p = (Ax^4 + Bx^3 + Cx^2 + Dx)e^x$$

for the trial solution. This shares no linearly independent parts with the homogeneous solution, so this is the proper form for the trial solution.

4. Use de Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

   Solution

   De Moivre's Theorem with $r = 1$ and $n = 3$ gives us that

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3,$$

   and simplifying the right hand side, we have

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

   Equating real and imaginary parts gives us

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

5. Prove the following properties of complex numbers, where a line over a complex number indicates its complex conjugate.

   (a) $\overline{z+w} = \overline{z} + \overline{w}$

   Solution

   Using rectangular coordinates, let $z = a + ib$, and let $w = c + id$. Then

$$\overline{z+w} = \overline{(a+c) + i(b+d)} = (a+c) - i(b+d) = (a - ib) + (c - id) = \overline{z} + \overline{w}.$$

   (b) $\overline{zw} = \overline{z}\,\overline{w}$

   Solution

   Using polar coordinates, let $z = re^{i\theta}$, and let $w = se^{i\phi}$. Then

$$\overline{zw} = \overline{rse^{i(\theta+\phi)}} = rse^{i(-\theta-\phi)} = re^{i(-\theta)}se^{i(-\phi)} = \overline{z}\,\overline{w}.$$

   (c) $\overline{z^n} = \overline{z}^n$

   Solution

   Using polar coordinates, let $z = re^{i\theta}$. Then

$$\overline{z^n} = \overline{r^n e^{in\theta}} = r^n e^{i(-n\theta)} = \left(re^{i(-\theta)}\right)^n = \overline{z}^n$$