Worksheet 11 Solutions, Math 1B
More Differential Equations

Monday, April 9, 2012

1. Classify the following differential equations according to order, homogeneity, linearity, and separability:

   (a) $y' = 3xy$
   (b) $y' = 3y^2$
   (c) $y'' = 3y + x$
   (d) $y''' + 3y' + xy = 2x^2$
   (e) $y'''y'' = 1$
   (f) $y' - y = 0$
   (g) $x^2 + 2x\cos(x) = y'$
   (h) $e^x y'' = e^{y''} x$
   (i) $x\ln(x)y' + y = xe^x$
   (j) $y' - y = (x+1)(x-1)y$

   Solution

	Order	Separable	Linear	Homogeneous
$y' = 3xy$	1	Yes	Yes	Yes
$y' = 3y^2$	1	Yes	No	N/A
$y'' = 3y + x$	2	N/A	Yes	No
$y''' + 3y' + xy = 2x^2$	3	N/A	Yes	No
$y'''y'' = 1$	3	N/A	No	N/A
$y' - y = 0$	1	Yes	Yes	Yes
$x^2 + 2x\cos(x) = y'$	1	Yes	Yes	No
$e^x y'' = e^{y''} x$	2	N/A	No	N/A
$x\ln(x)y' + y = xe^x$	1	No	Yes	No
$y' - y = (x+1)(x-1)y$	1	Yes	Yes	Yes

2. Find the general solution of the differential equation, then find a specific solution with the given conditions:

   (a) $y' = x + y, \quad y(0) = 2.$

   Solution
   This is a first-order linear differential equation which is not separable, so we apply the method of integrating factors. The standard form of the differential equation is

$$y' - y = x.$$

In order to simplify this form, we multiply by the integrating factor

$$I(x) = \exp\left(\int -1 \; dx\right) = e^{-x}$$

to get

$$y'e^{-x} - ye^{-x} = (ye^{-x})' = xe^{-x}.$$

From this form, we can simply integrate both sides, to get

$$ye^{-x} = \int xe^{-x} \; dx = -xe^{-x} - \int e^{-x} \; dx = -xe^{-x} + e^{-x} + C,$$

and dividing through by e^{-x}, we find the general solution

$$y = 1 - x + Ce^{x}.$$

The initial condition then gives us

$$y(0) = 2 = 1 - (0) + Ce^{0} = 1 + C,$$

and so $C = 1$. The solution of the initial value problem is then

$$y = 1 - x + e^{x}.$$

(b) $ty' + 2y = t^3, \quad t > 0, \quad y(1) = 0.$

Solution Idea
This is a first-order linear differential equation which is not separable, so the method of integrating factors is the appropriate solution technique. The standard form for the differential equation is

$$y' + (2/t)y = t^2,$$

so the integrating factor will be given by

$$I(x) = \exp\left(\int 2/t \; dt\right) = \exp\left(\ln |t|^2\right) = t^2.$$

(c) $xy' = y + x^2 \sin x, \quad y(\pi) = 0.$

Solution Idea
This is a first-order linear differential equation which is not separable, so the method of integrating factors is the appropriate solution technique. The standard form for the differential equation is

$$y' - (1/x)y = x \sin x,$$

so the integrating factor will be given by

$$I(x) = \exp\left(\int -1/x \; dx\right) = \exp\left(-\ln |x|\right) = \frac{1}{|x|}.$$

Since the function $P(x) = -1/x$ becomes undefined at $x = 0$, the solution should be restricted to the domain $x > 0$.

(d) $4y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.5.$

Solution
This is a second-order linear homogeneous differential equation with constant coefficients, so we may find a general solution using the method of characteristic equations. The characteristic equation of this differential equation is

$$4r^2 - 4r + 1 = 0,$$

and this gives us a single repeated root $r = 1/2$, and a general solution

$$y = C_1 e^{x/2} + C_2 x e^{x/2}.$$

The derivative of this function is

$$y' = C_1 e^{x/2}/2 + C_2 x e^{x/2}/2 + C_2 e^{x/2} = (C_1/2 + C_2)e^{x/2} + (C_2/2)xe^{x/2},$$

so evaluating at the initial conditions gives us that

$$\begin{cases} C_1 & = 1 \\ C_1/2 + C_2 & = -1.5 \end{cases},$$

or $C_1 = 1$, and $C_2 = -2$. Thus the solution to this initial value problem is

$$y = e^{x/2} - 2xe^{x/2}.$$

(e) $y'' - 2y' + 5y = 0, \quad y(\pi) = 0, \quad y'(\pi) = 2.$

Solution Idea
The characteristic equation of this differential equation is

$$r^2 - 2r + 5 = 0,$$

which has roots $r = 1 \pm 2i$, and so the general solution of the equation is

$$y = C_1 e^x \cos(2x) + C_2 e^x \sin(2x).$$

(f) $y'' + 2y' = 0, \quad y(0) = 1, \quad y(1) = 2.$

Solution Idea
The characteristic equation of this differential equation is

$$r^2 + 2r = 0,$$

which has roots $r = 0, -2$, and so the general solution of the equation is

$$y = C_1 + C_2 e^{-2x}.$$

3. Let L be a nonzero real number, and consider the boundary value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$. For the cases of $\lambda = 0$ and $\lambda < 0$, show that the problem has only the trivial solution $y = 0$. For the case of $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

Solution

The characteristic equation of this differential equation is given by

$$r^2 + \lambda = 0,$$

and this has a general solution whose form depends on the value of λ. For $\lambda < 0$, the form is

$$y = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x},$$

and applying the boundary conditions gives

$$\begin{cases} C_1 + C_2 & = 0 \\ C_1 e^{\sqrt{\lambda}L} + C_2 e^{-\sqrt{\lambda}L} & = 0 \end{cases}.$$

The first equation gives $C_1 = -C_2$, and using this relation in the second equation gives

$$C_2 e^{\sqrt{\lambda}L}\left(e^{-2\sqrt{\lambda}L} - 1\right) = 0,$$

which implies that we must have $C_2 = 0$, since λ and L are both assumed non-zero. In particular, this implies $C_1 = 0$, and so the only solution is $y = 0$.

In the case of $\lambda = 0$, the general solution has form

$$y = C_1 x + C_2,$$

and applying the boundary conditions gives $C_2 = 0$ and $LC_1 + C_2 = 0$, which implies $C_1 = C_2 = 0$ since $L \neq 0$. Thus in this case we also see that $y = 0$ is the only solution.

For $\lambda > 0$, the general solution is given by

$$y = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Applying the boundary conditions gives

$$\begin{cases} C_1 & = 0 \\ C_1 \cos(\sqrt{\lambda}L) + C_2 \sin(\sqrt{\lambda}L) & = 0 \end{cases},$$

and in particular, this implies that we must have

$$C_2 \sin(\sqrt{\lambda}L) = 0.$$

If $\sqrt{\lambda}L$ is equal to πk for some integer k, or

$$\lambda = \left(\frac{\pi}{L}\right)^2 k^2, \quad k \text{ an integer}$$

then C_2 may be arbitrary, and we find that there are non-trivial solutions. On the other hand, if this is not the case, then in order for the second condition to hold, we must have $C_2 = 0$, which means that there are no non-trivial solutions.