Worksheet 10 Solutions, Math 1B

Separable Differential Equations

Monday, April 2, 2012

1. Find the solution of the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \sqrt{Pt}$, that satisfies the initial condition $P(1) = 2$.

   Solution

   This differential equation is only defined when P and t are positive, so restrict our solutions to $t > 0$. This is a first-order separable differential equation so we write

   $$P^{-1/2}dP = t^{1/2}dt,$$

   and integrating both sides gives

   $$2\sqrt{P} = 2t^{3/2}/3 + C,$$

   and solving for P, we have

   $$P = (t^{3/2}/3 + C)^2$$

   Substituting the initial condition gives

   $$P(1) = 2 = ((1)^{3/2}/3 + C)^2,$$

   so we find $C = -1/3 \pm \sqrt{2}$, giving two solutions,

   $$P = \left(\frac{t^{3/2} - 1}{3} \pm \sqrt{2}\right)^2.$$

2. Find the solution of the differential equation $y' \tan x = a + y$, that satisfies the initial condition $y(\pi/3) = a$, $0 < x < \pi/2$.

   Solution Idea

   This first-order differential equation is non-separable, but linear. Written in standard form, the equation is

   $$y' - y/\tan x = a,$$

   and since it is non-separable, the most suitable method of solution is to use integrating factors. The integrating factor in this case is

   $$I(x) = \exp\left(\int -1/\tan(x)\right) = \exp\left(-\ln|\sin x|\right) = \frac{1}{|\sin x|},$$

   and in particular, this is $1/\sin(x)$ in the interval we are considering.

3. Solve the differential equation $xy' = y + xe^{y/x}$ by making the change of variable $v = y/x$.

   Solution Sketch

   The change of variable gives us $v' = y'/x - y/x^2$ by applying the product rule, and so if we rewrite the differential equation as

   $$x(y'/x - y/x^2) - e^{y/x} = 0,$$

we see that we can rewrite this as

$$xv' - e^v = 0,$$

or

$$v' = e^v/x.$$

This differential equation is separable, so we may solve by integrating to find $-e^{-v} = \ln|x| + C$, or

$$v = \ln\left(\frac{1}{\frac{1}{|x|} + C}\right).$$

Substituting $v = y/x$, this gives us a solution of

$$y = x\ln\left(\frac{1}{\frac{1}{|x|} + C}\right).$$

4. Sketch the direction field of the differential equation $y' = y + xy$, and use it to sketch a solution curve that passes through the point $(0, 1)$.

   Solution Idea

   Take a look at `http://en.wikipedia.org/wiki/Slope_field` for a good description and sample plots of slope fields of first-order differential equations.

5. According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x+R)^2}$$

   where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the constant of acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m(dv/dt)$, and so

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{mgR^2}{(x+R)^2}$$

   (a) Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R+h}}$$

   [*Hint:* By the Chain Rule, $m(dv/dt) = mv(dv/dx)$.]

   Solution Sketch

   Using $m(dv/dt) = mv(dv/dx)$ and thinking of v as a function of x, the above equation simplifies to $\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{v}\cdot\frac{gR^2}{(x+R)^2}$, which is a separable differential equation. Integrating yields

$$v = \sqrt{\frac{2gR^2}{x+R} + C}.$$

   The condition that the maximum height h is specified can be interpreted as an initial condition $v(h) = 0$, and this allows us to solve for C, finding

$$v = \sqrt{2gR^2\left(\frac{1}{R+x} - \frac{1}{R+h}\right)}.$$

Thus, letting $x = 0$, we find that the initial velocity v_0 is

$$v_0 = \sqrt{2gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)} = \sqrt{2gR^2\left(\frac{h}{R(R+h)}\right)} = \sqrt{\frac{2gRh}{R+h}}.$$

(b) Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the *escape velocity* for the earth.

Solution
The limit is given by

$$v_e = \lim_{h\to\infty} v_0 = \lim_{h\to\infty}\sqrt{\frac{2gRh}{R+h}} = \sqrt{\lim_{h\to\infty}\frac{2gR}{R/h+1}} = \sqrt{2gR}.$$

(c) Use $R = 3960$ miles and $g = 32\text{ft/s}^2$ to calculate v_e in feet per second.

Solution
We convert R into feet by the conversion

$$3960\text{mi}\cdot\frac{5280\text{ft}}{1\text{mi}} = 20908800\text{ ft},$$

and find that with these parameters,

$$v_e = \sqrt{2\cdot 32\text{ft/s}^2\cdot 20908800\text{ ft}} \approx 36581\text{ft/s}$$

This is very close to the known value of 36680ft/s, where the error in our answer is a result of error in the values we used for R and g.