Quiz 4 Solutions, Makeup

Friday, March 16, 2012

Name:

Student ID#:

Please place personal items under your seat.
No use of notes, texts, calculators, or fellow students is allowed.
Show all of your work in order to receive full credit.

1. Find the Maclaurin series of $\ln|1+x|$, and determine its radius of convergence.

   Solution

   The derivative of $\ln|1+x|$ is given by $1/(1+x)$, which has Maclaurin series

$$\sum_{n=0}^{\infty}(-1)^n x^n$$

This series is a geometric series, so it converges exactly when $|-x| = |x| < 1$, and so the series has radius of convergence 1. Integrating this term by term, we get a Maclaurin series for our original function (up to a constant) which has the same radius of convergence:

$$\int 1/(1+x) = \int \sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} + C$$

To compute the constant C, we substitute $x = 0$ into the series, and compare to $\ln|1+(0)|$:

$$\ln|1+(0)| = 0 = \sum_{n=0}^{\infty}\frac{(-1)^n (0)^{n+1}}{n+1} + C = C$$

Thus the appropriate constant of integration is $C = 0$, and we conclude that

$$\ln|1+x| = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},$$

with radius of convergence 1.

2. Evaluate the indefinite integral

$$\int \frac{\cos(\sqrt{x}) - 1}{x}\, dx$$

using power series. Your solution should be in the form of a power series.

Solution

The integrand is only defined for $x > 0$, so we restrict our considerations to positive x. We expand $\cos(\sqrt{x})$ as a power series, and simplify the integrand in this form:

$$\frac{\cos(\sqrt{x})}{x} = x^{-1}\left(\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} - 1\right) = x^{-1}\left(\sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(2n)!}$$

Integration can thus be done term by term over the power series, giving an indefinite integral of

$$\int \frac{\cos(\sqrt{x}) - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(2n)!} + C$$

3. Find the degree 2 Taylor approximation of $\sqrt{x}$ about the point $a = 4$. Find a bound for the error of this approximation at $x = 4.1$ using Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}\, |x - a|^{n+1}$$

Solution

We calculate the first and second derivatives of $f(x) = \sqrt{x}$ at $a = 4$ by

$$f'(x) = \frac{1}{2x^{1/2}} \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4x^{3/2}} \qquad f''(4) = -\frac{1}{32}$$

Then the degree 2 Taylor approximation of $\sqrt{x}$ about the point $a = 4$ is given by

$$P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)(x-a)^n}{n!} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64}.$$

The variable M in Taylor's inequality is a bound on the magnitude of the third derivative $f'''(x) = 3/(8x^{5/2})$ for $|x - a| < |4.1 - 4| = 0.1$. To find this bound, we notice that $f'''(x)$ is positive and decreasing on the interval $[3.9, 4.1]$, and therefore its magnitude in this interval is bounded by its value at the left endpoint, which is $3/(3.9)^{5/2}$.

Thus an error bound for the approximation is given by

$$|R_2(4.1)| \leq \frac{3/(3.9)^{5/2}}{3!}\, |4.1 - 4|^3 = \frac{(0.1)^3}{2 \cdot (3.9)^{5/2}}$$