Quiz 1 Solutions, Math 1B, Section 310

Friday, January 27, 2012

Name:

Student ID#:

Please place personal items under your seat.
No use of notes, texts, calculators, or fellow students is allowed.
Show all of your work in order to receive full credit.

Evaluate the following integrals (5 pts each):

1. $\int x^2 \cos(x)\, dx$

   Solution

   We apply integration by parts twice. First we use $u = x^2$ and $dv = \cos(x)dx$, which gives us $du = 2x\, dx$ and $v = \sin(x)$, and gives us an equivalent expression of

   $$x^2 \sin(x) - \int 2x \sin(x)\, dx.$$

   For the second time, we use $u = x$ and $dv = -\sin(x)\, dx$, which gives us $du = dx$ and $v = \cos(x)$. The resulting expression is

   $$x^2 \sin(x) + 2(x\cos(x) - \int \cos(x)\, dx) = (x^2 - 2)\sin(x) + 2x\cos(x) + C.$$

2. $\int \tan^3(x) \sec(x)\, dx$

   Solution

   We first apply the Pythagorean identity $1 + \tan^2(x) = \sec^2(x)$ to get

   $$\int (\sec^2(x) - 1)\tan(x)\sec(x)\, dx.$$

   From here we can simply substitute using $u = \sec(x)$, to get

   $$\int (u^2 - 1)\, du = u^3/3 - u + C = \sec^3(x)/3 - \sec(x) + C.$$

3. $\int_0^a \sqrt{a^2 - x^2}\, dx$

Solution

We begin by making a sine-type substitution $x = a\sin(\theta)$, where θ is chosen between $-\pi/2$ and $\pi/2$. Under this substitution, the integral becomes

$$\int_0^{\pi/2} \sqrt{a^2 - a^2\sin(\theta)}a\cos(\theta)\, d\theta,$$

and using the Pythagorean identity $1 - \sin^2(\theta) = \cos^2(\theta)$, we see that this is just

$$a^2 \int_0^{\pi/2} \cos^2(\theta)\, d\theta.$$

To solve this, we need to make use of the half-angle formula

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

to rewrite this integral as

$$\begin{aligned} a^2/2 \int_0^{\pi/2} (1 + \cos(2\theta))\, d\theta &= a^2/2(\theta + \sin(2\theta)/2)\Big|_0^{\pi/2} \\ &= a^2/2(\pi/2 + 0 - 0 + 0) = \pi a^2/4. \end{aligned}$$

If the bounds of integration were not translated with the substitution $x = a\sin(\theta)$, the reverse substitution would also require the use of the double-angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ along with the usual triangle construction for combining trig and inverse trig functions.

Notice that this solution makes sense, as the integral is over the upper-right quadrant of a circle of radius a centered at the origin, and so its value should be equal to a quarter of the circle's area.