Quiz 4 Solutions, Section 309

Friday, March 16, 2012

Name:

Student ID#:

Please place personal items under your seat.
No use of notes, texts, calculators, or fellow students is allowed.
Show all of your work in order to receive full credit.

1. Find the Maclaurin series of $\cos(\sqrt{2x})$, and determine its radius of convergence.

   Solution

   The function $f(x) = \cos(\sqrt{2x})$ is only defined for $x \geq 0$, but we may determine a Maclaurin series for f on this domain by substituting $\sqrt{2x}$ into the Maclaurin series for cosine:

   $$\cos(\sqrt{2x}) = \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{2x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{(2n)!}$$

   Since the radius of convergence for cosine is infinite, the series converges for any value of $\sqrt{2x}$. In particular, this means that the series converges for any positive value of x, and since any finite radius of convergence would imply divergence for sufficiently large positive x, we can conclude that the radius of convergence of this series is infinite.

2. Evaluate the indefinite integral

   $$\int \frac{e^{-x^2} - 1}{x^2}\, dx$$

   using power series. Your solution should be in the form of a power series.

   Solution

   We expand e^{-x^2} as a power series, and simplify the integrand in this form:

   $$\frac{e^{-x^2} - 1}{x^2} = x^{-2}\left(\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} - 1\right) = x^{-2}\left(\sum_{n=1}^{\infty} \frac{(-x^2)^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{n!}$$

   Integration can thus be done term by term over the power series, giving an indefinite integral of

   $$\int \frac{e^{-x^2} - 1}{x^2}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)n!} + C$$

3. Find the degree 2 Taylor approximation of $\ln(x)$ about the point $a = 2$. Find a bound for the error of this approximation at $x = 1.8$ using Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}\,|x-a|^{n+1}$$

Solution

We calculate the first and second derivatives of $f(x) = \ln(x)$ at $a = 2$ by

$$\begin{aligned} f'(x) &= \frac{1}{x} & \qquad f'(2) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{x^2} & \qquad f''(2) &= -\frac{1}{4} \end{aligned}$$

Then the degree 2 Taylor approximation of $\ln(x)$ about the point $a = 2$ is given by

$$P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)(x-a)^n}{n!} = \ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8}.$$

The variable M in Taylor's inequality is a bound on the magnitude of the third derivative $f'''(x) = 2/x^3$ for $|x-a| < |1.8-2| = 0.2$. To find this bound, we notice that $f'''(x)$ is positive and decreasing on the interval $[1.8, 2.2]$, and therefore its magnitude in this interval is bounded by its value at the left endpoint, which is $2/(1.8)^3$.

Thus an error bound for the approximation is given by

$$|R_2(1.8)| \leq \frac{2/(1.8)^3}{3!}\,|1.8-2|^3 = \frac{(0.2)^3}{3 \cdot (1.8)^3}$$