Math 1A Quiz Ch. 5

November 25, 2013

1. (4 points) Using sigma notation and a limit, write the definition of the definite integral $\int_a^b f(x)\,dx$ in terms of Riemann sums.

   Solution

   The integral is defined to be equal to

   $$\lim_{n\to\infty} \frac{b-a}{n} \cdot \sum_{i=0}^{n-1} f\left(x_i^*\right),$$

   where x_i^* is, in each case, any point in the interval $[a+i(b-a)/n, a+(i+1)(b-a)/n]$, assuming that this limit exists and is independent of the choice of points x_i^*.

2. (6 points) Compute the definite integral:

   $$\int_0^1 (x^2+1)e^{-x}\,dx$$

   Solution

   We first have

   $$\begin{aligned}\int_0^1 (x^2+1)e^{-x}\,dx &= \int_0^1 x^2e^{-x}\,dx + \int_0^1 e^{-x}\,dx \\ &= \int_0^1 x^2e^{-x}\,dx - e^{-x}\Big|_0^1 = \int_0^1 x^2e^{-x}\,dx + 1 - 1/e\end{aligned}$$

   Now for the remaining integral, we use integration by parts. Letting $u = x^2$ and $v' = e^{-x}$, we get $v = -e^{-x}$ and $u' = 2x$, giving us

   $$\int_0^1 x^2e^{-x}\,dx = \left(-x^2e^{-x}\right)\Big|_0^1 - \int_0^1 -2xe^{-x}\,dx = -1/e + 2\int_0^1 xe^{-x}\,dx$$

   For the last integral, we again use integration by parts. Letting $u = x$ and $v' = e^{-x}$, we get $v = -e^{-x}$ and $u' = 1$, so

   $$\begin{aligned}\int_0^1 xe^{-x}\,dx &= \left(-xe^{-x}\right)\Big|_0^1 - \int_0^1 -e^{-x}\,dx \\ &= -1/e - \left(e^{-x}\right)\Big|_0^1 = -1/e - 1/e + 1 = 1 - 2/e\end{aligned}$$

   Thus the final value for the integral is

   $$(-1/e + 2(1-2/e)) + 1 - 1/e = 3 - 6/e.$$

3. (6 points) Compute the indefinite integral:

$$\int \frac{2^t}{2^t+3}\,dt$$

Solution

We use a u-substitution, with $u = 2^t + 3$. Then with this choice of u, we have $du = \ln 2 \cdot 2^t\,dt$, so we have

$$\int \frac{2^t}{2^t+3}\,dt = \int \frac{1}{\ln 2}\frac{\ln 2 \cdot 2^t}{2^t+3}\,dt = \frac{1}{\ln 2}\int \frac{1}{u}\,du = \frac{\ln|u|}{\ln 2} + C = \frac{\ln(2^t+3)}{\ln 2}.$$

(We can take $2^t + 3$ out of the absolute values in the end because it is always positive.)

4. (6 points) Compute the indefinite integral:

$$\int \tan^{-1}(x)\,dx$$

Solution

We use integration by parts, with $u = \tan^{-1}(x)$ and $v' = 1$. Then $u' = 1/(1+x^2)$ and $v = x$, so we get

$$\int \tan^{-1}(x)\,dx = x\tan^{-x}(x) - \int \frac{x}{1+x^2}\,dx.$$

Using a u-substitution of $u = 1 + x^2$ on the remaining integral gives us $du = 2x\,dx$, so we see that

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{2}\frac{2x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln|u|}{2} + C = \frac{\ln(1+x^2)}{2} + C,$$

where we can take $1+x^2$ out of the absolute values in the last step because it is always positive. Thus we see that the final value for the integral is

$$\int \tan^{-1}(x)\,dx = x\tan^{-x}(x) - \frac{\ln(x^2+1)}{2} + C.$$

5. (10 points) Find the derivative of the function:

$$F(x) = \int_x^{x^2} e^{t^2}\, dt$$

Solution

Denote by $G(t)$ an antiderivative of e^{t^2}. In particular, $G'(t) = e^{t^2}$. Then we have that

$$F(x) = \int_x^{x^2} e^{t^2}\, dt = G(x^2) - G(x),$$

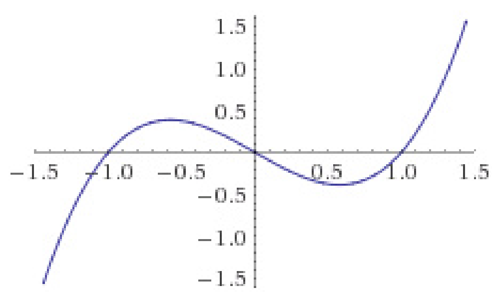
so the derivative is

$$F'(x) = G'(x^2) \cdot 2x - G'(x) = 2xe^{x^4} - e^{x^2}.$$

6. (6 points) Find the area enclosed between the curve $y = x^3 - x$ and the x-axis.

Solution

The graph of $y = x^3 - x$ has roots at $x = -1, 0, 1$, and looks like the following:



Thus the area enclosed between the curve $y = x^3 - x$ and the x-axis is given by twice the area between the graph and the x-axis over the interval $[-1, 0]$, which, since y is positive on this interval, is given by

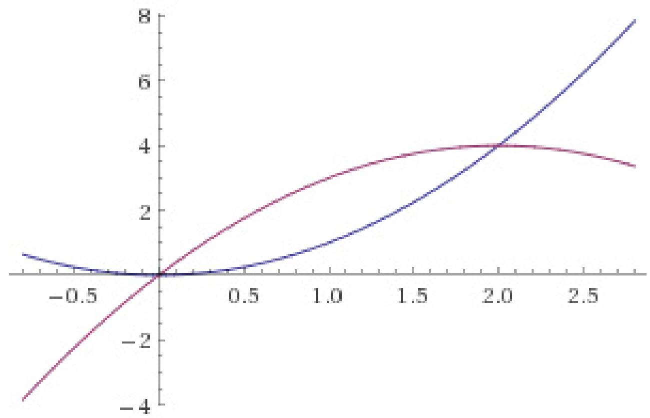
$$\int_{-1}^{0} x^3 - x\, dx.$$

This is just $2 \cdot (x^4/4 - x^2/2)\big|_{-1}^{0} = 1/2$.

7. (6 points) Find the area between the curves $y = x^2$ and $y = 4x - x^2$.

Solution

The graphs of $y = x^2$ and $y = 4x - x^2$ look like the following:



Their intersections occur when $x^2 = 4x - x^2$, or $x^2 - 2x = x(x-2) = 0$, which gives $x = 0$ and $x = 2$. Then since $4x - x^2$ is the larger of the two functions on the interval $(0, 2)$ the area between the two curves is

$$\int_0^2 (4x - x^2) - (x^2)\,dx = 2\int_0^2 2x - x^2\,dx = 2(x^2 - x^3/3)\big|_0^2 = 2(4 - 8/3) = 8/3.$$

8. (6 points) Use an integral to compute the volume of a circular cone with base of radius r and height h.

Solution

The volume of a solid is given by the integral in a particular direction of its cross-sectional area. In particular, if we position the cone so that its tip is at the origin and the central axis goes in the positive x-direction, we notice that the cross-sections perpendicular to the x-axis are just circles whose radii are proportional to x.

Since we know that at the base of the cone the radius of the circle is r, we see that at a given x-coordinate in $[0, h]$, the radius of the cross-section is $x/h \cdot r$. Thus the cross-sectional area is $\pi(x/h \cdot r)^2$. This means that the volume of the cone is given by

$$\int_0^h \pi x^2 r^2/h^2\,dx = \pi r^2/h^2 \cdot \int_0^h x^2\,dx = \pi r^2/h^2 \cdot (x^3/3)\big|_0^h = \frac{1}{3}\pi r^2 h.$$