Math 1A Quiz Ch. 4

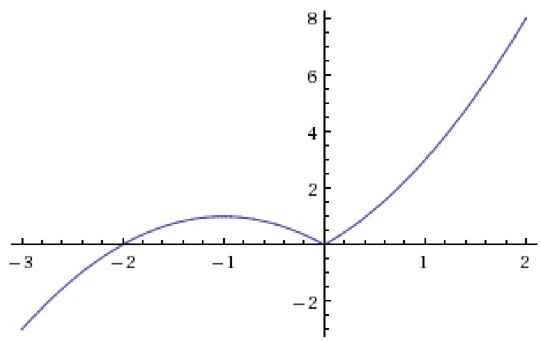
November 4, 2013

1. (8 pts) Define a "critical number" of a function f. Sketch the graph of the function $f(x) = |x| \cdot (x+2)$, and find its critical numbers. Find the absolute maximum and absolute minimum values of f on the interval $[-2, 1]$.

   Solution

   A critical number of a function f is a number x in the domain of f such that either $f'(x) = 0$ or f' is undefined at x.

   The graph of f looks like:



   In particular, f is not differentiable at $x = 0$, so this is a critical number of f. Additionally, for $x < 0$ we have $|x| = -x$, and for $x > 0$ we have $|x| = x$, so that means that

$$f(x) = \begin{cases} x^2 + 2x & x \geq 0 \\ -x^2 - 2x, & x < 0 \end{cases}, \quad \text{and} \quad f'(x) = \begin{cases} 2x + 2, & x \geq 0 \\ -2x - 2, & x < 0 \end{cases}.$$

   Thus we have one additional critical number, $x = -1$ where $f'(x) = 0$.

   To find the absolute max and min of f, we need to check the value of f at its critical numbers and the boundaries of the domain, that is, $x = -2$, $x = -1$, $x = 0$ and $x = 1$. This gives us

$$f(-2) = 0, \quad f(-1) = 1, \quad f(0) = 0, \quad \text{and} \quad f(1) = 3.$$

   Thus we have that the absolute minimum of f is 0, achieved at $x = -2$ and $x = 0$, and the absolute maximum of f is 3 at $x = 1$.

2. (12 pts) Show that the equation $x^3 + e^x = 0$ has exactly one real root.

Solution

The function $f(x) = x^3 + e^x$ is differentiable anywhere since it is the sum of a polynomial and an exponential function, so it is also continuous everywhere. In particular, we have that $f(-1) = -1 + 1/e < 0$, and $f(0) = 1 > 0$, so by the intermediate value theorem, since f is continuous on $[-1, 0]$, we know that there must be a point $c \in (-1, 0)$ such that $f(c) = 0$. Thus f has at least one real root.

Now suppose by way of contradiction that f has at least two real roots, call them $a < b$. Then f is continuous on $[a, b]$ and differentiable on (a, b), so by the mean value theorem, we know that there is a number $d \in (a, b)$ such that $f'(d) = (f(b) - f(a))/(b - a) = 0$. But we know that $f'(x) = 3x^2 + e^x > 0$ for all real x, so it can't be the case that $f'(x) = 0$ for any real x. This gives us a contradiction, so we can conclude that our assumption that f has at least two real roots must have been false.

Thus we have shown that f has at least one real root, and that it also has at most one real root, and so we see that f has exactly one real root.

3. (10 pts) Let $f(x) = x^3 - 12x + 2$.

   (a) Find the intervals of increase or decrease of f.

   (b) Find the local maximum and minimum values.

   (c) Find the intervals of concavity and the inflection points.

   (d) Use the information you found to sketch the graph of f. (Don't worry about finding the precise roots of f.)

Solution

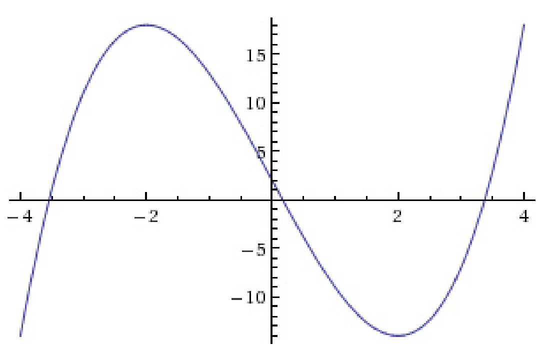
We have $f'(x) = 3x^2 - 12$, and $f''(x) = 6x$.

f is increasing on intervals where $f'(x) > 0$ and decreasing on intervals where $f'(x) < 0$, so that means that f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$, and is decreasing on the interval $(-2, 2)$.

Since f is differentiable everywhere, if f has a local extreme value at x, we must have $f'(x) = 0$. This only occurs at $x = \pm 2$. Looking at the signs of the first derivative shows us that $x = -2$ is a local maximum and $x = 2$ is a local minimum, but we can get this information directly by looking at the sign of the second derivative: $f''(-2) = -12 < 0$ and $f''(2) = 12 > 0$.

Finally, f is concave up on intervals where $f''(x) > 0$ and concave down on intervals where $f''(x) < 0$, and has an inflection point where the second derivative changes sign (and f is continuous). Thus f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$, and $f''(x)$ changes sign at $x = 0$ which means that f has an inflection point at $x = 0$.

We also notice that $f(0) = 2$, and $f(x)$ tends to $-\infty$ as $x \to -\infty$, and to $+\infty$ as $x \to +\infty$. Thus the graph of f looks like:



4. (6 pts) Use L'Hospital's Rule to evaluate the limit:

$$\lim_{x \to 0^+} x^{\sqrt{x}}$$

Solution

Naively plugging in values to this limit gives an expression of the form 0^0, which is not an indeterminate form of the correct form to use L'Hospital's Rule, so we need to rewrite the expression. We have

$$x^{\sqrt{x}} = e^{\ln(x)\sqrt{x}}$$

so since the exponential function is continuous everywhere, if $\ln(x)\sqrt{x}$ has a finite limit L, then x^{sqrtx} has finite limit e^L.

Taking limits as $x \to 0^+$ of this expression gives us $-\infty \cdot 0$, which is now a multiplicative indeterminate form. Rewriting it as $\ln(x)/x^{-1/2}$, we get the indeterminate form $-\infty/\infty$, so now we can use L'Hospital's Rule.

Looking at the limit of the ratio of derivatives, we have

$$\lim_{x \to 0^+} \frac{x^{-1}}{-1/2 \cdot x^{-3/2}} = \lim_{x \to 0^+} -\sqrt{x}/2 = 0,$$

so by L'Hospital's Rule, we know that $\lim_{x \to 0^+} \ln(x)\sqrt{x}$ exists, and also has value 0. Thus we conclude that $\lim_{x \to 0^+} x^{\sqrt{x}} = e^1 = 1$.

5. (6 pts) A cylindrical can without a top is made to contain a particular volume V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

Solution

Let r be the radius of the cylinder, and let h be the height of the cylinder. Both parameters must be positive values. Then the surface area of the can is roughly proportional to the volume of metal used to make the can, and is equal to $2\pi rh$, the area of the cylinderical sides, plus πr^2, the area of the circular bottom, or

$$A = 2\pi rh + \pi r^2.$$

Additionally, if we want to require that the can contains volume V, then we have $V = \pi r^2 h$, which gives us

$$h = V/(\pi r^2).$$

Plugging into the expression for area, we find that, as a function of radius, we must have $A(r) = 2V/r+\pi r^2$. Taking a derivative, we find $A'(r) = -2V/r^2+2\pi r$, so A has a single critical number $r = \sqrt[3]{V/\pi}$. Taking a second derivative, we find $A''(r) = 4V/r^3+2\pi$, so the second derivative at $r = \sqrt[3]{V/\pi}$ is $6\pi > 0$, meaning that this gives a local minimum of A.

We also see that A tends to $+\infty$ as $r \to 0^+$ and as $r \to +\infty$, so we conclude that $r = \sqrt[3]{V/\pi}$ is the radius that will minimize the value of A. This value of r corresponds to a height $h = \sqrt[3]{V/\pi}$.