Math 1A Quiz Ch. 3
Solutions

October 23, 2013

1. (4 pts) Find the derivative of $f\left(\frac{g(x)}{h(x)} + x^2\right)$ with respect to x.

   Solution

   We apply the chain rule, along with the quotient rule, to find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(f\left(\frac{g(x)}{h(x)} + x^2\right)\right) \\
&= f'\left(\frac{g(x)}{h(x)} + x^2\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{h(x)} + x^2\right) \\
&= f'\left(\frac{g(x)}{h(x)} + x^2\right) \cdot \left(\frac{h(x)g'(x) - g(x)h'(x)}{(g(x))^2} + 2x\right)
\end{aligned}$$

2. (4 pts) Find $g'(x)$ where $g(x) = \sqrt{\cos(\sin^2 x)}$.

   Solution

   We apply the chain rule, and use the derivatives for sin and cos to find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(\sin^2 x)^{\frac{1}{2}}\right) \\
&= \frac{1}{2\cos(\sin^2 x)^{\frac{1}{2}}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\big(\cos(\sin^2 x)\big) \\
&= \frac{1}{2\cos(\sin^2 x)^{\frac{1}{2}}} \cdot \big(-\sin(\sin^2 x)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\big(\sin^2 x\big) \\
&= \frac{1}{2\cos(\sin^2 x)^{\frac{1}{2}}} \cdot \big(-\sin(\sin^2 x)\big) \cdot 2\sin(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\
&= \frac{1}{2\cos(\sin^2 x)^{\frac{1}{2}}} \cdot \big(-\sin(\sin^2 x)\big) \cdot 2\sin(x) \cdot \cos(x)
\end{aligned}$$

3. (4 pts) A particle moves on a line so that its coordinate at time t is $y = -5t^2 + 10t + \sqrt{2}$, $t \geq 0$. Find the velocity and acceleration functions.

Solution

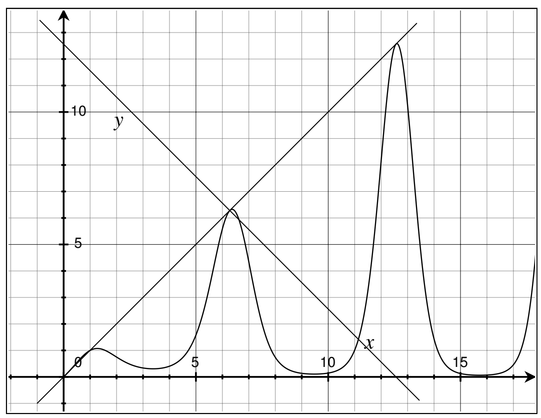
If a particle has position y, then its velocity is given by y' and its acceleration is given by y''. Thus we have

$$v(t) = y' = -10t + 10,$$

and

$$a(t) = y'' = -10.$$

4. (16 pts) Find the equations of the tangent line and the normal line to the curve $y = x^{\cos x}$ at $x = 2\pi$. Draw the lines in the picture of the graph below. *Hint: to find dy/dx you can use logarithmic differentiation, or you can write y as $e^{(something)}$.*



Solution

First we compute the derivative of y by writing $y = x^{\cos x} = e^{\ln x \cos x}$. Then we have, using the chain rule and the product rule,

$$y' = \frac{\mathrm{d}}{\mathrm{d}x} e^{\ln x \cos x} = e^{\ln x \cos x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} (\ln x \cos x) = x^{\cos x} \cdot \left(\frac{\cos x}{x} - \ln x \sin x \right).$$

Alternatively, we could have used logarithmic differentiation by taking logarithms of both sides of the equation and applying implicit differentation.

Now to find the tangent line to the graph at $x = 2\pi$, we need the value of y at 2π and also the value of y'. Plugging in the values gives us $y = 2\pi$ and $y' = 1$. Then using point-slope form we see that the tangent line is given by the equation $y - 2\pi = 1 \cdot (x - 2\pi)$, or $y = x$.

To find the normal line, we note that if the tangent line at a point has slope $m \neq 0$, then the normal line at this point has slope $-1/m$. In our case, at $x = 2\pi$, the slope is $-1/1 = -1$. Thus the normal line is given by the equation $y - 2\pi = -1 \cdot (x - 2\pi)$, or $y = -x + 4\pi$.

5. (10 pts) A cylindrical tank with radius 5 m is being filled with water at a rate of $3\text{m}^3/\text{min}$. How fast is the height of the water increasing? *Remember to define your variables!*

   Solution

   Let h represent the height of the water in the tank, and let V represent the volume of water in the tank.

   Then the volume and the height are related by the equation $V = \pi r^2 h$ for the volume of a cylinder, which gives us

   $$V(t) = \pi(5\text{m})^2 h(t).$$

   Now we can use implicit differentation to see that $V'(t) = 25\pi\text{m}^2 h'(t)$, or

   $$h'(t) = V'(t)/25\pi\ \text{m}^2.$$

   Since we know that $V'(t) = 3\ \text{m}^3/\text{min}$, this gives us that the water is rising at a rate of $h'(t) = (3/25\pi)\ \text{m}/\text{min}$.

6. (10 pts) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

   Solution

   Let h represent the height of the water in the cup, r the radius of the circular top of the water level, and V the volume of water in the cup.

   The ratio of radius over height is constant for cones with the same angle, so because the paper cup has radius 3 cm and height 10 cm, the ratio r/h is equal to 3 cm over 10 cm, or $r/h = 3/10$. This gives us that $r = 3/10 \cdot h$.

   Now we can relate the volume of water in the cup to the height and radius by the equation for the volume of a cone, $V = (1/3)\pi r^2 h$. In particular, using our equation for r in terms of h, this gives us

   $$V(t) = (1/3)\pi \cdot (3/10 \cdot h(t))^2 \cdot h(t) = (3\pi/100) \cdot h(t)^3.$$

   Using implicit differentiation on this relation gives us $V'(t) = (9\pi/100) \cdot h(t)^2 h'(t)$, or

   $$h'(t) = V'(t)/(9\pi/100) \cdot h(t)^2.$$

   When $h(t) = 5$ cm, the rate of change of volume is of course $V'(t) = 2\ \text{cm}^3/\text{s}$ (since this value is constant), so substituting these values into the above relation tells us that the water level is rising at a rate of $h'(t) = 8/(9\pi)$ cm/s.

7. (10 pt) A ladder 13 ft long rests against a vertical wall. If the top of the ladder slides down the wall at a rate of 1 ft/s, how fast is the bottom of the ladder sliding along the floor when the top of the ladder is 12 ft from the floor?

   Solution

   Let x be the distance from the wall to the bottom of the ladder, and let y be the distance from the top of the ladder to the floor. Then since the ladder forms a right triangle with the wall and the floor, the pythagorean identity gives us that

$$x(t)^2 + y(t)^2 = 169 \text{ m}^2.$$

   Using implicit differentiation gives us the further relation $2x(t)x'(t) = 2y(t)y'(t) = 0 \text{ m}^2/\text{s}$, so we have

$$x'(t) = -y(t)y'(t)/x(t).$$

   Now when the top of the ladder is 12 ft from the floor, the ladder forms a 5-12-13 right triangle, so the bottom of the ladder is 5 ft from the wall, giving us values of 12 ft and 5 ft for y and x respectively. Further, since the ladder is sliding down the wall at a constant rate of 1 ft/s, that means that y' always has value -1. Substituting these values into the equation for x' gives us that the ladder is sliding along the floor at a rate of $x'(t) = 12/5$ ft/s.