Name:

Math 110, Section 105, Quiz 9
Wednesday, October 25, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ Any eigenspace of a linear operator T is T-cyclic.

b. ________ If T is a linear operator on a vector space V, then any T-invariant subspace of V is T-cyclic.

c. ________ Let T be a linear operator on a finite-dimensional vector space V with dimension n. Then there exists a polynomial $g(t)$ of degree at most $n-1$ such that $T^n = g(T)$.

Solution. F F T

Exercise. Let $V = \mathbb{R}^4$ and let T be a linear operator on V given by $T(x, y, z, w) = (x+y, y-z, x+z, x+w)$. Find a basis of the T-cyclic subspace generated by the vector $v = e_1$, and prove it is a basis.

Solution. Let W denote the T-cyclic subspace generated by e_1. If k is the smallest number such that $\{e_1, T(e_1), \ldots, T^k(e_1)\}$ is linearly dependent, then the set $\{e_1, T(e_1), \ldots, T^{k-1}(e_1)\}$ forms a basis of W. In particular, we compute

$$\begin{aligned} T(1,0,0,0) &= (1,0,1,1) \\ T(1,0,1,1) &= (1,-1,2,2) \\ T(1,-1,2,2) &= (0,-3,3,3) \end{aligned}$$

Notice that $(0,-3,3,3) = 3(1,-1,2,2) - 3(1,0,1,1)$, so the set $\{e_1, \ldots, T^3(e_1)\}$ is linearly dependent. If $\{e_1, \ldots, T^2(e_1)\}$ is linearly independent, then it forms a basis. Row reducing, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus all three columns are pivot columns, and we see that the set is linearly independent. Thus $\beta = \{(1,0,0,0), (1,0,1,1), (1,-1,2,2)\}$ gives a basis of W.