Math 110, Section 105, Quiz 7
Wednesday, October 11, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary column operations.

b. ________ Cramer's Rule gives a solution to any linear system $Ax = b$ of n equations in n unknowns.

c. ________ The function $\det : M_{n\times n}(F) \to F$ given by taking the determinant of a matrix is a linear transformation.

Solution. F F F

Exercise. Compute the determinant of the following 5×5 matrix. (*Hint:* Think rook placements.)

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. Because there are so many zeros in A, only a few of the $5! = 120$ total rook placements result in a nonzero term in the summation formula for $\det(A)$. In particular, since rows 1 and 5 have only a single nonzero term, these entries must always be used in a nonzero rook placement. Now that only three additional rooks need to be placed, we see that there are exactly two nonzero rook placements in total, one for each choice of a rook in column 1:

$$\begin{pmatrix} & * & & & \\ * & & & & \\ & & & * & \\ & & * & & \\ & & & & * \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} & * & & & \\ & & * & & \\ * & & & & \\ & & & * & \\ & & & & * \end{pmatrix}$$

Computing inversions by counting the number of $*$'s which are arranged in a lower-left to upper-right orientation, we see that both of these arrangements have two inversions. Thus we can compute

$$\det(A) = (-1)^2 \cdot (2 \cdot 1 \cdot 2 \cdot 1 \cdot 1) + (-1)^2 \cdot (2 \cdot 3 \cdot 1 \cdot 2 \cdot 1) = 4 + 12 = 16$$