Math 110, Section 101, Quiz 2
Wednesday, September 6, 2017

This quiz will be graded out of 15 points; the True/False question is worth 3 points, and the exercise is worth 12 points. Please read the instructions carefully, and explain your work.

True or False. Mark the following statements as either true or false, or leave a blank if you don't know. A correct answer is worth +1 point, a blank is worth 0 points, and an incorrect answer is worth -1 points, so be smart about guessing!

a. ________ Any intersection of subspaces of a vector space V is a subspace of V.

b. ________ The diagonal matrices are a subset of the upper triangular matrices in $M_{n\times n}(F)$.

c. ________ It is acceptable to attend a discussion section different from the one you're enrolled in if your current enrollment conflicts with another class.

Solution. T T F

✷

Exercise. Let W_1 and W_2 be the subspaces of $\mathbb{R}^3$ given by

$$W_1 = \{(a, b, b) : a, b \in \mathbb{R}\}, \quad W_2 = \{(c, 0, -c) : c \in \mathbb{R}\}$$

Prove that $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

Solution. In order for W_1 and W_2 to form a direct sum, we need their intersection to be just the zero element, and we need their sum to be the entire space $\mathbb{R}^3$.

For the first, note that if $x \in W_1 \cap W_2$, then there are real numbers a and b such that $x = (a, b, b)$, and there is a real number c such that $x = (c, 0, -c)$. But then, by comparing the coordinates, we see that we must have $b = 0$, so $-c = b = 0$ means that $c = 0$, and $a = c$ means that $a = 0$. Thus we have $x = (0, 0, 0)$, so that $W_1 \cap W_2$ consists of only the zero element.

To see that $W_1 + W_2 = \mathbb{R}^3$, note that since $W_1, W_2 \leq \mathbb{R}^3$ we automatically have that $W_1 + W_2 \subseteq \mathbb{R}^3$, so we need only show that $\mathbb{R}^3 \subseteq W_1 + W_2$, i.e. that any element of $\mathbb{R}^3$ can be written as a sum of an element of W_1 with an element of W_2. To see this, note that for any $\mathbf{x} = (x, y, z)$, we can choose $b = y$, $c = b - z = y - z$, and $a = x - c = x - y + z$. Then we get

$$(a, b, b) + (c, 0, -c) = (a + c, b, b - c) = ((x - y + z) + (y - z), y, y - (y - z)) = (x, y, z)$$

Thus an arbitrary vector $\mathbf{x}$ can be written as a sum of the appropriate form, so $\mathbb{R}^3 \subseteq W_1 + W_2$.