Math 110, Discussion Section Exercises
Sections 101, 103, and 105
Wednesday, September 13, 2017

The following exercises were explored in discussion section, where we worked on them in small groups. For the best learning experience, make sure to spend at least 10 or 15 minutes working on each problem *before* looking at the solution.

Exercise 1. Let $\beta = \{e^z, e^{iz}, e^{-z}, e^{-iz}\} \in \mathcal{F}(\mathbb{C}, \mathbb{C})$, and let $V = \operatorname{Span}(\beta)$, where in particular β is an ordered basis of V. Define the function $T : V \to V$ by $T : f \mapsto \frac{\mathrm{d}}{\mathrm{d}z} f$.

a) Prove that T is linear. Is T one-to-one? Onto?

b) Compute the matrix $[T]_\beta$.

c) If $\gamma = \{\sin z, \cos z, \sinh z, \cosh z\}$, compute $[T]_\gamma$.

d) Also compute $[\mathrm{Id}]_\beta^\gamma$ and $[T]_\beta^\gamma$. (Here Id denotes the "identity map", which does nothing to its inputs: $\mathrm{Id}(f) = f$.

Exercise 2. Let β and γ be two ordered bases of $\mathbb{R}^2$ given by

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \gamma = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

and consider the correspondence (isomorphism) between $L(\mathbb{R}^2, \mathbb{R}^2)$ and $M_{2\times 2}(\mathbb{R})$ given by the map $\psi : T \mapsto [T]_\beta^\gamma$.

a) What is the image of the identity map Id under this correspondence? (In other words, what is the matrix given by $\psi(\mathrm{Id})$?)

b) What linear transformations in $L(\mathbb{R}^2, \mathbb{R}^2)$ correspond with the upper triangular matrices in $M_{2\times 2}(\mathbb{R})$

Exercise 3. Find a vector space V and a linear transformation $T : V \to V$ such that T is one-to-one but not onto. For most vector spaces we will work with in this class, this is not possible; what property of V allows this situation to occur? Find the kernel and image of T.

Solutions

1. Concerning complex exponential functions: if you haven't encountered these in a previous class, don't worry! They aren't very scary. The main things you need to know are:

- They satisfy the same rules of multiplication and exponentiation that normal exponentials do:
$$e^{\alpha z} \cdot e^{\beta w} = e^{\alpha z + \beta w}, \text{ and } (e^{\alpha z})^{\gamma} = e^{\gamma \cdot \alpha z}$$
- They can be translated to a usual complex form using trig functions:
$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \text{ for } \theta \in \mathbb{R}$$
so
$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$
- Derivatives and integrals work as usual:
$$\frac{\mathrm{d}}{\mathrm{d}z} e^{\alpha z} = \alpha e^{\alpha z}, \text{ and } \int e^{\alpha z}\,\mathrm{d}z = \frac{1}{\alpha} e^{\alpha z} + C$$

If hyperbolic trig functions are freaking you out, have no fear:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

1a. T is linear because the derivative function is linear. Any linear combination of differentiable functions such as the exponentials in β is differentiable, so if $f, g \in V$ and $\alpha \in \mathbb{C}$, then

$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g), \text{ and } T(\alpha f) = (\alpha f)' = \alpha(f') = \alpha T(f)$$

Thus T is linear. V is finite dimensional, so T is one-to-one iff T is onto, and in this case it is both rather than neither. To see that T is onto, note that

$$T(e^z) = e^z, T(-ie^{iz}) = e^{iz}, T(-e^{-z}) = e^{-z}, \text{ and } T(ie^{-iz}) = e^{-iz}$$

Thus $\beta \subseteq \mathrm{Im}(T)$, and so we see that the image of T is the entire space V.

1b. By computing the derivatives above, we see that the image of each basis element in β is just a scalar times that basis element, which means that the matrix $[T]_\beta$ is diagonal, namely

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

1c. The derivatives of the trig functions in γ are given by

$$T(\sin(z)) = \cos(z), T(\cos(z)) = -\sin(z), T(\sinh(z)) = \cosh(z), \text{ and } T(\cosh(z)) = \sinh(z)$$

Thus the matrix representing each of these basis functions as a linear combination of the same basis functions (by giving a column with the coefficients of the linear transformation) is

$$[T]_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1d. The matrix representing the identity map Id is just the matrix representing the basis elements of β in terms of the basis elements of γ. In particular, we have

$$\begin{aligned} e^z &= \sinh(z) + \cosh(z) & \qquad e^{iz} &= i\sin(z) + \cos(z) \\ e^{-z} &= -\sinh(z) + \cosh(z) & \qquad e^{-iz} &= -i\sin(z) + \cos(z) \end{aligned}$$

Thus using the coefficients of each linear combination as the columns of a matrix, we see that

$$[\mathrm{Id}]_\beta^\gamma = \begin{pmatrix} 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Because differentiation only results in multiplication by the constant in the exponential for each $f \in \beta$, the corresponding matrix for T is the same as the above, except each column is multiplied by the appropriate constant.

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & i & 0 & -i \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

2a. To find the matrix $\phi(\mathrm{Id}) = [\mathrm{Id}]_\beta^\gamma$, we compute the decomposition of each vector in β as a linear combination of the vectors in γ. Namely

$$(1,0) = 1\cdot(1,1) + 1\cdot(0,-1), \quad (0,1) = 0\cdot(1,1) - 1\cdot(0,-1)$$

Thus

$$\phi(\mathrm{Id}) = [\mathrm{Id}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

(Note however, that it is an accident that the columns of this matrix are the vectors of γ; this does not usually occur.)

2b. The linear transformations T whose matrix $[T]_\beta^\gamma$ is upper diagonal can be reconstructed by looking at the images of each of the vectors in the standard basis β. Namely, if

$$[T]_\beta^\gamma = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

for some constants $a, b, c \in \mathbb{R}$, then $T((1,0)) = a\cdot(1,1) = (a,a)$, and $T((0,1)) = b\cdot(1,1) + c\cdot(0,-1) = (b, b-c)$.

In particular, we can then write

$$\begin{aligned} T((x,y)) &= T(x\cdot(1,0)+y\cdot(0,1)) \\ &= x\cdot T((1,0))+y\cdot T((0,1)) \\ &= x\cdot(a,a)+y\cdot(b,b-c) \\ &= (ax+by, ax+(b-c)y) \end{aligned}$$

Thus the upper triangular matrices correspond with the linear transformations of the form given above for some constants $a, b, c \in \mathbb{R}$. Restated, these are the linear functions such that the coefficient of x is the same in each of the two coordinates of the function expression.

3. A classic example of a vector space and linear transformation is the space V of infinite sequences of real numbers:

$$V = \{(a_0, a_1, a_2, a_3, \ldots) : a_i \in \mathbb{R} \text{ for each } i\}$$

along with the linear transformation T called the "right shift" operator, which is given by

$$T : (a_0, a_1, a_2, a_3, \ldots) \mapsto (0, a_0, a_1, a_2, a_3, \ldots)$$

(Sanity check: What are the addition and scalar multiplication of V as a vector space?) It is easy to verify that T is a linear transformation. To see that it is one-to-one, note that if we had two points $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ which map to the same sequence when applying T, then we would have

$$(0, a_0, a_1, a_2, \ldots) = (0, b_0, b_1, b_2, \ldots)$$

which implies that $a_i = b_i$ for every i, and thus that the two points we started with are the same. However, T is clearly not onto, since its image does not contain (for instance) the sequence $(1, 0, 0, 0, \ldots)$.

The main property that is necessary for this situation to occur is that V is infinite-dimensional, i.e. it does not contain a finite set of vectors which form a basis. In general if V is assumed to be finite-dimensional, then we know that a linear transformation $T : V \to V$ is one-to-one if and only if it is onto, and properties like this are part of the reason why we tend to focus on finite-dimensional vector space.

Finally, note that the kernel of T is trivial (if the shifted sequence is all zeros, then the original sequence must have been all zeros as well), and the image of T is the set of sequences $(a_0, a_1, a_2, \ldots)$ with $a_0 = 0$.

Followup questions: Can you find a similar transformation which is onto but not one-to-one? Can you find similar maps on the space $P(\mathbb{R})$ of all polynomials with real coefficients? Hint: Think about Problem 1 from these exercises.